THE PUPIL'S ADVOCATE*

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Once upon a time this Association appointed a committee to consider the rival claims of courses in mathematics and of professional courses in education in a fifth year program for prospective teachers of secondary mathematics. This committee reported that it naturally favored mastery of subject matter, but that it was bound to admit that the continued existence and growth of instruction in educational psychology, in the principles of teaching, and in allied subjects indicated some merit in those areas also. It recognized that instruction in "how to teach" was often feeble, and that therefore it was easy to see why such instruction was generally scorned by the academic mind. Nevertheless this committee took the position that where there was smoke there must also be fire, and decided to recommend for the fifth year an almost equal division between the claims of mathematics and professional training in education. They presumed, I suppose, that a prospective teacher whose training included both mathematics and some knowledge of the learning process would blend both ingredients in an effective teaching procedure. You can read the report of this committee in this MONTHLY, volume 42, May, 1935, page 263.

If this committee of your own Association could take such a position concerning secondary mathematics in 1935, it seems to me reasonable to expect that many members of this Association would be sympathetic to the idea that prospective teachers of freshman and sophomore mathematics in our colleges today should blend with their mastery of subject matter some knowledge of how the human mind acquires new ideas, how it schools itself in new procedures, and how the acquisition of these new ideas and procedures is helped or hurt by the attitudes and emotions that accompany them. I do not propose to tell you how you might alter the training of prospective college teachers from its present pattern so as to include some of the ideas on "how to teach" that I believe would be helpful. Three things, however, I will say: first, that the important ideas and attitudes that I have in mind can be put into such compact form that they can be picked up in relatively short time; second, that even a small amount of properly concentrated instruction in how to teach could result in great improvement in teaching in freshman and sophomore classes; and third, that it is futile to advocate the inclusion of such instruction in the training of prospective college teachers until college departments of mathematics see the worth of it and show that they want it. Once they decide that they want it, it will not be difficult to provide it. Until they decide that they want it, the obstacles to providing it will be numerous.

Instead, therefore, of wasting my time on proposals for modifying the present training program, I will endeavor to show the sorts of changes in instruction you might expect if prospective college teachers were brought to consider the

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human mind and how it learns. If you don't like the changes that I predict, you need only sit tight. If, in time, you come to the point where you can see advantage to the cause of college mathematics in making certain changes, you will find a way to do so.

The changes that I believe would result from giving more thought to the pupil and to the manner in which he learns would necessarily involve some changes in courses of study and in text books. These changes, as I envisage them, would be small, but their effect on the pupil would be great. The changes that interest me most all stem from one idea, namely, proper motivation of the subject-matter. This, broadly conceived, includes as a corollary the idea of withholding certain details of rigorous exposition until the pupil appreciates their import. When I say "withhold," I do not mean "suppress." However, I do mean something that you will probably not enjoy, even though you come in time to accept the idea. For it implies some disruption and some dislocation of the purely logical presentation of a subject. It implies beginning some subjects in the middle and working both ways from that first plunge. After all, has not much of mathematics been developed in this same way, beginning on the ground floor and then advancing to higher levels while new and better foundations were tucked in underneath?

The first attack on a subject, instead of being straightforward and logically ordered, would in some instances be fragmentary and messy. It would require more artful exposition than merely proceeding in logical order; for, in the interest of truth, we should wish to indicate where some of the truth is withheld, and to indicate at times also the general nature of the truth that we withhold. It implies some repetition also, in the form of orderly recapitulation; because, once the pupil appreciates the inadequacies of the initial attack, which I have described as fragmentary and messy, we do indeed want him to see the argument set down in order, with as much attention to rigor as he can accept. This explains what I had in mind by disruption and dislocation of the usual logical presentation.

The reason that I suggest a dual procedure of this sort with respect to certain topics is that the psychological approach to a subject is often quite different from the logical. I advocate the psychological approach in order to win and to hold the pupil's attention; I advocate a subsequent logical treatment because that is the goal of all our effort. I believe that if we proceed in this way the pupil will be more likely to accept and to heed the nice distinctions that interest us, and that he too will become interested in these fine points when he appreciates their significance. I think that our present approach causes us to lose many pupils, some by physical withdrawal, and others by mental and spiritual withdrawal. In one or two instances (overplaying limit of function and continuity at the start of the calculus, and overprotection against certain familiar pathological functions later on) the severely logical approach of some teachers, who seem to the pupil to personify mathematics itself, and of some authors seems to me to savor overmuch of the "Damn you, love me" technique that brings the young

lover to the front pages of our newspapers, but indicates to most of us that that is no way to woo.

The reason for my prediction above, that you will not enjoy the prospect of a dual treatment of certain topics, is that what I have called "merely proceeding in logical order" already demands exposition of a high order, as you well know; the even greater demands on the teacher's powers of exposition to provide an extra higglety-pigglety treatment out of deference to the pupil's psyche will really test the depth of your devotion to the pupil and your interest in connecting him in some way with mathematics. In case these demands seem to you to be just too much, please recall that my suggestions are aimed at only a few crucial topics in mathematics and that I am counting on great changes on the part of the pupil in return for only a few changes on our part.

Please understand that I think I appreciate the conflict in interest and attitude that are inherent in the teacher's task of bringing together the severely logical subject of mathematics, that requires thinking, just thinking all unadorned, and the mind of the human animal, that finds pure thinking difficult and is easily diverted to other methods of solving problems that to greater or less degree involve the emotions. We meet this same conflict in acute form when we attempt to use congenial exploratory and inductive methods to lure secondary pupils to contemplate the less congenial deductive aspects of demonstrative geometry. The problem there is how to teach deduction inductively.

I am sure that many college teachers regard it as altogether obvious that in beginning a new subject, say the infinitesimal calculus, they should build a good foundation and that the pupil for his part should review any bits of earlier instruction that will be needed in the new. The introductory chapter of text book after text book is built on that obvious pattern. It defines important ideas such as sequence, limit of a sequence, variable, function, limit of a function, continuous function, and announces that, if two variables have limits, the limit of the sum of the two variables is the sum of their two limits. Often the author of the text book shows that he recognizes that the pupil will consider such an approach altogether dreary. He suggests that the pupil skip these introductory ideas but remember where he can find them if ever he needs them. So the pupil is led without introduction, or by a highly repellent introduction that is worse than no introduction, to contemplate the limit of the ratio of two variables each of which is tending toward zero. And now observe the trouble we are in. The pupil who has been conducted through the introductory chapter has met the necessary ideas, but has acquired a distaste for them. The pupil who has been protected against an unfortunate emotional set toward the subject by being allowed to skip the introductory chapter finds now that he must grub around in it after all. It is quite possible that a sympathetic teacher, and there are many such, will not probe too embarrassingly at the start into the pupil's ideas concerning the limit of delta-x and the limit of delta-y. The teacher may deem it better not to molest a pupil who happens to think of delta-x as assuming only rational values as it tends toward zero. But however sympathetic the teacher,

he must face this fundamental conflict: that you can't begin the calculus without emphasizing the idea of limit, and that, whether or not the pupil has studied the introductory chapter, you can't count on his having an accurate notion of limit, ready for immediate use. To my mind, one of the most important outcomes of instruction in the infinitesimal calculus is the pupil's ultimate acquisition of sound ideas concerning variable and limit, and his appreciation of the vital role that these ideas play in establishing methods of calculating those quantities from geometry and physics that are the chief concern of the calculus and that the pupil recognizes as important and interesting. But the pupil will require time and patient handling if he is ultimately to acquire these ideas in the form that we desire. My plea is not for a cheapening of the ultimate goal of our instruction. I urge merely a compassionate approach that would save as many souls as possible at the beginning and along the way. At the beginning, therefore, I favor a method that relies heavily on the pupil's intuition and only gradually reveals to him that his intuitive ideas will bear examination and revision. I believe that I am correct in asserting that the usual formal introductory approach to the calculus is a direct affront to the pupil's intuition and for that reason is psychologically bad. I repeat, I think that such an approach antagonizes many pupils unnecessarily; that it loses us many pupils who could be saved to understand and enjoy mathematics; and that by saving them we could render a distinct service to the cause of mathematics.

I believe that the way to save them is easy; that it requires very little time and effort; and would necessitate very little change in courses of study or in the printed page. It may indeed require a considerable change in attitude at certain points. And this brings me back to my basic idea, the proper motivation of the subject-matter. The word motivation implies respect for the pupil's interests and attitudes; and it is good strategy for us to respect them, for interests and attitudes provide the drive to action. If you will trust my expressed intention that all the treasured ideas of limit, continuous function, and so on are eventually to be saved; and if my hope is realized that more pupils than formerly will be saved to appreciate these ideas; then why not introduce the pupil to the calculus by first showing him one or two examples of what the calculus can do and how it does it? Give him some reason for studying the calculus; appeal to his interests; stimulate in him a spirit of inquiry. For example, we can invent some plausible reason for wishing to know the maximum value of $5+6x-x^2$ as x varies—say, the number of inches $f(x)$ of growth of pole beans in terms of the number of inches x of rain in June—and show the bare technique by which this maximum value is computed. No mention of variable secant, no deltas, no limit; nothing but the rigmarole of passing from $5+6x-x^2$ to $6-2x$, of attaching to this latter expression the idea of slope of tangent, and then eliciting from the pupil the connection between relative maximum and zero slope, and allowing him to finish the computation himself then and there. The pupil's earlier instruction in algebra has probably already given him an idea or two concerning the critical point of a quadratic function against which he can check this new

technique of the calculus. And then, in another few minutes, another example involving a cubic function set equal to y, where our new technique, called differentiation, produces y', the slope of the tangent. And then, with equal abandon, a third example in which the pupil employs this new technique to pass from $s=16t^2$ to $s'=32t$, and is given immediately a brief picture of the connection between slope of tangent and rate of change of a function in general. In this last example it is conceivable that the pupil can suggest that rate in this case can be interpreted as velocity; at least the teacher will invite him to try to make this association. And then the teacher can suggest the possibility of reversing this operation so that, given the relation between velocity and time, one can infer a relation between distance and time. There is no attempt here at explanation; the sole intent is to give a quick preview of some of the ideas that will occupy the pupil in his study of the calculus; to show him quickly and with very simple examples how the calculus is linked to geometry and physics; to contrast the two main operations of the calculus; and, above all else, to make the subject appear in prospect to be easy. The pupil is not expected to understand much of what he has been shown. The object is rather to show him that it involves concepts already familiar to him and is the sort of thing that he can understand. A further object is to establish a feeling of confidence between pupil and teacher; for at this same time the teacher must say that certain steps in reasoning that support the simple techniques just exhibited will seem to the pupil to involve nice distinctions, the need for which is not immediately apparent, but the teacher understands this and does not expect immediate and complete acceptance of every new idea on the part of the pupil; that allowance will be made for a period of incubation. I am aware that this sounds a bit childish. But if so, is it not because we commonly forget how close the adult is to the child in matters of this sort? In brief, I am suggesting that if we defer somewhat to the psychological child in the pupil before us, we increase the likelihood that he will accept the adult mathematics that we set before him. Now, just before you start to tear this suggestion to pieces, allow me to insist that I am not discussing at this point the teaching of the calculus, and that I do not want my remarks with respect to $5+6x-x^2$ and thereafter to be taken as a sample of how I would teach those topics in the calculus. We are not in the calculus yet; only on the doorstep. I am discussing the introduction to the calculus; and not so much how to introduce the idea of limit as how to introduce the pupil. The word calculus still means to me all that it has ever implied at the beginning level with respect to the idea of limit.

I sympathize with the desire to parade at the beginning of a book on the calculus the materials of the present introductory chapter. From the point of view of the learner, however, I incline to favor the dissemination of this material throughout the book as need for it arises. That is one of the disruptions that I mentioned earlier. Then, considering the learner and also the teacher, I am inclined to favor a collection of all this material for ready reference in the appendix. And that is one of the dislocations that I mentioned.

If only the introduction could show the pupil why he is studying the calculus, and lead him at once to the derivative as usually explained, and to exercises and problems involving the derivative, I think the pupil would be better served and better pleased. The exposition of the derivative would suffer no change. The pupil would continue to be as naïve as formerly about variable and limit, but his morale would be better. Instead of being given the idea that he was doing less than expected and was getting off to a bad start because he had not entirely understood the necessary preliminaries about variable and limit, he would be given the idea that he was doing all that was expected at this time, that it was adequate for the present, but that he must expect very soon to acquire greater sensitivity to the techniques involved and to the ideas on which they rest. Most books require the pupil to apply the fundamental definition of the derivative to a series of ten or more expressions like $4x^2+3$ and $3x^2-5x+4$. I would allow the pupil to consider the earlier ones in this list as aimed chiefly at giving him facility in the technique of translating $f(x+\Delta x)-f(x)$ into terms appropriate to a given exercise. And then in subsequent exercises I would expect the teacher to be more demanding of the pupil with respect to the limit idea. Just how demanding depends at this stage on how much of this the pupil can accept.

It is possible, of course, to take the point of view that if we defer always to the pupil, he will stage a sit-down strike and baulk us in our attempt to reach our goal. Actually, I think the human animal reacts quite differently: that when pushed and crowded he does indeed sit-down and baulk; but when held responsible for displaying his own powers under conditions that give him a fair chance to do so, he surprises us by doing more than we expect. I believe it is all too easy to find examples of pupils who sit passively in our class-rooms and respond only briefly to our best efforts. I have an idea that if we should pay more attention to how pupils learn, we could discover ways of converting our best efforts into even better and more successful efforts.

In similar vein, would it not be worth while to devote a few minutes at the beginning of infinite series to explain why we interrupt the steady course of the calculus to include this topic? And why we make so much fuss over the matter of convergence? It is a rare book that lets the pupil in on the secret that $x \tan x$ cannot be integrated in terms of elementary functions; that admits that the strange and brief assortment of curves presented as exercises under length of arc are the best that can be assembled, again because of difficulties with integration; or that discloses why certain differential equations are offered as exercises and why others, much like them, are not offered. I believe that it would cheer the pupil immeasurably to learn that author and teacher are less than omnipotent, that there are things they cannot do. Why not tell him? In this connection I ask why we list in a table of integrals only those expressions that we can integrate in terms of elementary functions; why not include some of the closely related functions that cannot be handled so readily, with a simple notation to

that effect? I believe that the pupil would take hold of infinite series much better if we told him at the outset that one reason for introducing the study of infinite series into the calculus is to circumvent this barrier of non-integrability. And then, instead of proceeding to a formal exposition of the first ideas on infinite series—handing it out cold, as the pupil would say—my own inclination would be to lead the pupil to do a bit of experimenting at this point. I suggest that he try to employ the first few terms of the binomial expansion in an effort to evaluate $(2+1)^{-2}$; $(1+2)^{-2}$; $(4+1)^{1/2}$; $(1+4)^{1/2}$; $1/(1+x)$ for $x=\frac{1}{2}$; and $1/(1+x)$ for $x=2$. Nothing would be said at this time about the validity of employing the binomial expansion in this way. Further, I would suggest that the pupil integrate from $x=0$ to $x=h$ the first six terms of the expansion of $1/(1+x)$ in order to approximate the integral $\int_0^h dx/(1+x)$ and try in this way to find $\log(1+\frac{1}{2})$ and $\log(1+2)$. And I wouldn't be above asking him to contrast the expansion of $1/(1+x)$ when x equals 1 with the value of his series for $\log(1+h)$ when h equals 1. I should expect that the pupil would see some point now in considering whether a given series converges or not.

With respect to convergence, there is some danger of piling up ingenious examples that run far beyond any use that the pupil will make of this important idea. In this respect this topic is reminiscent of factoring in algebra. With respect to both topics it can be said, and truly, that the fussy cases are introduced to test the pupil's grasp of basic principles. If we recall why we introduced series and why we decided to inquire into the matter of convergence, it will be easy to determine when the pupil has had enough. Certain series serve admirably to test the pupil's grasp of the idea of limit of a sequence and deserve consideration for that reason if for no other. By now the pupil has progressed far beyond the point where we need be solicitous about his attitude toward the idea of limit; our concern at this stage is to seek occasions that will cause him to revert to the idea and test his mastery of it.

Another opportunity to appeal to the pupil's interest is afforded by the hyperbolic functions, but I do not recall any text that seizes this opportunity. To me it seems almost unavoidable that the pupil, comparing the integrals

$$\int_0^x \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

and

$$\int_0^x \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log_e\left(\frac{x}{a} + \frac{\sqrt{a^2 + x^2}}{a}\right)$$

and their geometric counterparts right-triangle-plus-circular-sector and right-triangle-plus-hyperbolic-sector, as in Figure 1, would be compelled by the striking analogy to ruminate on the result of relabeling $\log_e(x/a + \sqrt{a^2+x^2}/a)$, which

we shall call u for short, as $\sinh^{-1} x/a$. He would then be curious as to what $\sinh u$,

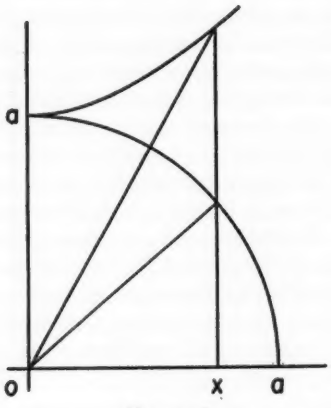

FIGURE 1

equal to x/a, might look like. He can easily satisfy his curiosity about x/a by converting

$$\log_e\left(\frac{x}{a}+\frac{\sqrt{a^2+x^2}}{a}\right)=u$$

into

$$\frac{x}{a}+\frac{\sqrt{a^2+x^2}}{a}=e^u,$$

which yields

$$\frac{x}{a}=\frac{e^u-e^{-u}}{2}=\sinh u.$$

Similarly, by comparing the negative integral

$$\int_a^x \sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{a^2}{2}\cdot\frac{\pi}{2}$$

$$=\frac{x}{2}\sqrt{a^2-x^2}-\frac{a^2}{2}\cos^{-1}\frac{x}{a}$$

and the positive integral

$$\int_a^x \sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log_e\left(\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right)$$

and their geometric counterparts right-triangle-minus-(larger)circular-sector and right-triangle-minus-(smaller)hyperbolic-sector, as shown in Figure 2, the

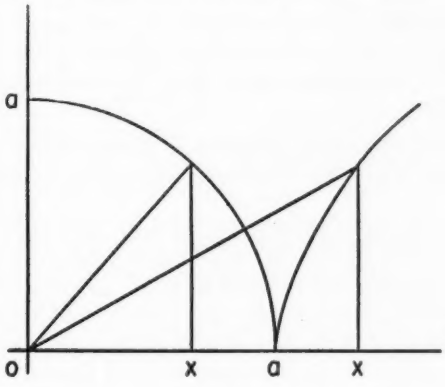

FIGURE 2

pupil would be tempted by the analogy to try the result of writing

$$\log_e\left(\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right)=v=\cosh^{-1}\frac{x}{a}.$$

From

$$\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}=e^v$$

he gets

$$\frac{x}{a}=\frac{e^v+e^{-v}}{2}=\cosh v.$$

Giving further rein to his fancy he invents the hyperbolic tangent, the derivatives of the hyperbolic sine, cosine, and tangent, and pokes around for the anti-hyperbolic tangent. He gets this by twisting

$$\tanh v=\frac{e^v-e^{-v}}{e^v+e^{-v}}=w$$

into

$$e^{2v}=\frac{1+w}{1-w},$$

from which he derives

$$v = \tanh^{-1} w = \frac{1}{2} \log \frac{1+w}{1-w}.$$

This brings the integrals

$$\int \frac{dx}{1+x^2} = \tan^{-1} x \quad \text{and} \quad \int \frac{dx}{1-x^2} = \frac{1}{2} \log \frac{1+x}{1-x}$$

much closer together; and the pupil will find further support for the results of this fanciful inquiry when he works out the series developments of these functions.

This takes a bit more time than just "handing it out cold"; but I believe that it pays dividends in the long run. My object is to arouse the pupil's interest, to insist that he sees the meaning of what he does, and to lead him to go on his own whenever possible. Neither you nor I want him to accept procedures blindly, trusting the word of teacher or of the book. But sometimes we get behind in our program, possibly just because we have listened long and patiently to pupils' difficulties, objections, doubts, and inquiries. Remembering the demands of the syllabus of the course we are teaching, we are tempted then to stifle questions and proceed with the program. Sometimes, of course, we must do this. But the pupil knows whether our attitude in general is to prefer his interest to the demands of the syllabus, or whether we tend to give the syllabus the right of way. What good does it do if the teacher completes the course on time, but loses the pupil on the way?

I never cease to wonder how some instructors can proceed blandly on with the discussion of topics *B* and *C* when the pupils in front of them are not yet satisfied with important details of topic *A*. Cannot such an instructor imagine the discontent such treatment breeds in the pupil? Does he not see that the pupil who is still dissatisfied with respect to topic *A* is ripe to be taught; that he is emotionally set to inquire, to test one alternative after another; in short, is emotionally primed really to study? Here is an opportunity not to be missed. It does not mean necessarily more debate in class at this time; it may be better to suggest that the pupil—it is almost time to call him the "student"—follow this or that line of inquiry and report on it at the next meeting. Almost anything will serve that pays attention to the pupil at such a time. It is important that the instructor remember on his part to follow up the pupil's inquiry at their next meeting.

Ignoring a pupil who is emotionally set to learn tends to turn him against the instructor and against the course. What is the syllabus for, anyhow, that it is more important than the pupil?

What does the pupil bring to your class? Some proficiency; some misconceptions; some interest; some fear. It is the teacher's task to increase this proficiency; to replace misconception with proper understanding; to hold whatever interest the pupil may have had and to add to this interest. The teacher can do

little of any of these, however, unless he dispels the pupil's fear. Well, what is the pupil afraid of? He is afraid that he will not succeed with worded problems; that he will not really understand the "why" and "what for" of the various mathematical maneuvers he is expected to perform. He inclines to seek the security of a sure rule of thumb. You know where that will lead him! Consequently the teacher's chief task is to dispel these doubts; to create an atmosphere of confidence, in which the pupil will expose his ignorance and misconceptions freely, and will even welcome the teacher's probing to uncover other weaknesses that the pupil has not confessed. In such an atmosphere doubtful points in the teacher's exposition and in the book will be cleared up. Exercises that encourage the pupil to discover mathematics himself in his own imperfect way are an aid to this end; and so are exercises that allow for plenty of learning by actual doing. Admittedly these things all take time. But it is amazing how rapidly a pupil can acquire the information we want him to acquire if only we take pains to give him the proper emotional set first of all. Teachers in general know that they ought to provide an effective motive for learning; that they ought to allow some time for the pupil to discover some of mathematics for himself; and that there is an efficacy in the pupil's own doing that outweighs that which is done for him. But there are schedules to be maintained; the class must cover the ground. In many cases, if only the teacher would begin by concentrating on the pupil and ignoring the syllabus, he would discover that the pupil finally will rescue the syllabus for himself. There is much to be said for a slow beginning, to save as many souls as possible; those who survive can tolerate cheerfully a more rapid pace at the end.

It is my belief that many teachers and authors of textbooks recognize the merit of these ideas. Why, then, are they not more generally adopted? I will hazard the guess that it is partly because of ignorance and partly because of fear. Our knowledge of mathematics is relatively sure; our knowledge of human beings is uncertain. Then, too, it is more respectable to be concerned with the subject than with the pupil. If the teacher teaches, or the author writes, with his attention chiefly on the pupil, he risks suspicion as one who does not "know his stuff." I think that there are ways by which one who is really master of his subject can protect his scholarly reputation by adroit qualifications and occasional foot-notes. Some authors have shown that they can do this. I wish that more would follow suit.

The environment in which the college teacher works is concerned chiefly with his subject. What forces act on him in behalf of the pupil? If only it could be generally understood that every author has heard of the so-called "pathological functions" and was guaranteed free of attack because he seemed to ignore them, I think the pupil would be pleased and would progress more rapidly to the point where he too could join safely in the sacred rites surrounding them. I recognize the important part that exceptional cases and famous puzzles have played in the development of mathematics. But we can never attain a completely rigorous presentation on the adult level even; we must draw the line

somewhere. You can see that I think that we commonly draw it with too little regard for the pupil.

In this plea for greater consideration of the pupil's point of view I have drawn my illustrations from the calculus and have said nothing with respect to the subjects that normally precede it. You will not be surprised, however, at my opinion that, from the point of view of the subject, the best way to begin analytic geometry is in terms of directed line segments and their projections on the axes; but that, from the point of view of the pupil, such a beginning is unfortunate. My own inclination would be to rely at first on the pupil's intuition concerning the general validity of x_2-x_1 as the projection on the x-axis of the directed line segment P_1P_2. I am fully aware that the generality of the algebraic processes that we employ in analytic geometry forms the very heart of the subject; but I prefer to insinuate this idea gradually as the course develops. I believe that direct frontal assault causes the loss of too many of the participants; that repeated small forays at different times and places will bring more participants to the desired goal. Furthermore, directed line segments do not quite live up to the high hopes we have of them. It is a bit of an art, indeed one of the trickiest aspects of analytic geometry for the beginnner, to know when directed line segments are to be respected and when they can be safely ignored. I might even say "profitably ignored," having in mind the possible loss of the inner loop of the limaçon $r=3-4\cos\theta$ by too stuffy insistence on positive r's only. You know when you think it well to consider length, area, and volume as unsigned quantities and when you would be relieved of this restriction in order to win some important generalization. To the pupil the rules of the game are not so clear.

Again, in trigonometry, the majority of college teachers who write texts in this subject seem to think that if they begin with general angle they can clean up on the elementary functions once and for all. In my opinion the pupil would prefer to be introduced to the elementary functions as they appear in right triangles, with some subsequent extension short of the general angle to accommodate obtuse-angled triangles. I am aware that the style has turned a bit from the solution of triangles and that analytic trigonometry is more in favor. I think I know why, and I agree. I mention, merely, that I think that the approach through triangles means more to the pupil; and I would not ignore the pupil's point of view.

In summary, I advocate giving the game back to the pupil. Let him find out a few things for himself in the trial-and-error method by which we all learn. I would not be so solicitous for his safety in the mathematical world. A good soaking may teach him more than he would ever learn if he started out with full equipment of rubbers, raincoat, and umbrella. Actually this metaphor does not express my whole meaning, because I want him to return home after the first shower and get more equipment, once he sees what he is in for. Of course, it would make things easier at home if he would start out in the first place with all the equipment he might possibly require, even including crampons, and perhaps

water and lunch, in case he is marooned on a glazed highway and cannot move. That is the way some recent textbooks impress me. They care for every emergency in advance. That way everything is neat and orderly. But most of it is done by the guide; little initiative is left to the pupil.

Changing the metaphor, I would say that on the whole our mathematical house is in good order and that we take pride in keeping it so. I am not asking for a thorough house-cleaning. Quite the contrary. The house is spick and span now and needs no house-cleaning. I think, however, that it does need a little room that the children can call all their own, where a certain amount of disorder is tolerated, provided that from time to time the children are expected to tidy up the room according to their own standards of what is neat and orderly. They will not lack examples of adult standards of neatness and order in all other parts of the house.

A CLASS OF NON-DESARGUESIAN PLANE GEOMETRIES*

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1. Introduction. F. R. Moulton has shown† that Desargues' theorem is not a consequence of certain of Hilbert's axioms‡ by exhibiting a simple non-desarguesian plane geometry fulfilling these axioms. The points of this geometry are points of the euclidean plane, the lines of non-positive slope are euclidean lines, and the lines of positive slope are euclidean broken lines, broken at the x-axis so that the slope in the upper half-plane is a positive constant (not unity) times the slope in the lower half-plane.§

The present paper generalizes Moulton's paper in two ways. First, his geometry is generalized by consideration of the broken-line plane geometries defined by an arbitrary functional relationship between the upper and lower half-plane angles. The class of functions which define projective plane geometries is determined, and the non-desarguesian projective planes are found. Second, the non-desarguesian property of Moulton's geometry is generalized to show that Desargues' theorem is one of a large class of theorems which are not valid in each of the non-desarguesian projective planes.

2. The broken-line projective geometries. The points of our geometries consist of points of a real euclidean plane, and the lines include a euclidean axis creating an upper and lower plane and the euclidean parallels to this axis. The remaining lines of our geometries are the broken lines meeting the axis at an an-

* The author is indebted to Professor R. H. Bruck for his encouragement and advice in the preparation of this paper.

† A simple non-desarguesian plane geometry, Trans. Amer. Math. Soc., 3, 1902, pp. 192–195.

‡ Axioms I 1–2, II, III, IV 1–5, V. See Hilbert, Foundations of Geometry, 1902, chap. I.

§ The slope ratio is usually taken as two to one. See M. Pasch and M. Dehn, Vorlesungen über Neuere Geometrie, 1926, p. 256.

gle θ in the upper plane and an angle ϕ in the lower plane, as shown in Figure 1. Evidently, it is only necessary to consider a relationship between the angles θ and ϕ of these lines to be given by a single-valued function $\phi = f(\theta)$, with a single-valued inverse, such that $0 \leqq \theta \leqq \pi$, $0 \leqq \phi \leqq \pi$, $f(0) = 0$, and $f(\pi) = \pi$. Ordinary euclidean geometry is thus defined by $f(\theta) \equiv \theta$, $0 \leqq \theta \leqq \pi$.

As plane axioms of projective geometry we take Veblen and Young's assumptions of alignment,* which are as follows:

A1. If A and B are distinct points, there is at least one line on both A and B.

A2. If A and B are distinct points, there is not more than one line on both A and B.

A3. If A, B, C are points not all on the same line, and D and E ($D \neq E$) are points such that B, C, D are on a line and C, A, E are on a line, there is a point F such that A, B, F are on a line and also D, E, F are on a line.

Since all the geometries satisfying these assumptions are desired, we seek a necessary and sufficient condition on the function $\phi = f(\theta)$ so that the geometries defined by the functions satisfy these assumptions. For this purpose, define S as the class of single-valued strictly monotonic increasing continuous functions in

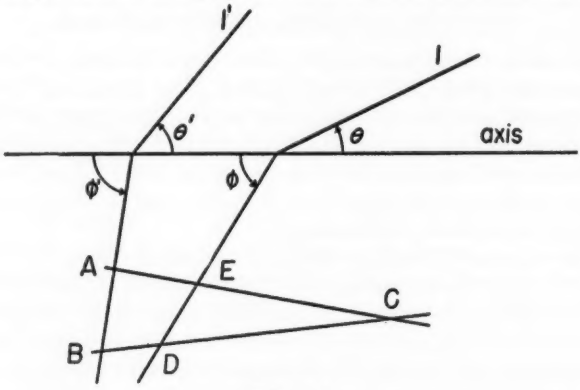

FIGURE 1

the closed interval $(0, \pi)$, with endpoints $(0, 0)$ and (π, π). We will prove

THEOREM 1. *A necessary and sufficient condition that a function $\phi = f(\theta)$ define a geometry satisfying the assumptions of alignment is that $\phi = f(\theta)$ be a member of class S.*

First, assume that a function $\phi = f(\theta)$ defines a geometry satisfying the assumptions of alignment. By definition, the function has the endpoints $(0, 0)$ and (π, π), and the function, and its inverse, are single-valued in the closed interval $(0, \pi)$. The strictly increasing monotonicity can be shown by considering lines l and l' (Fig. 1) for two values θ and θ', with $\theta' > \theta$. Lines CEA and CDB can be drawn as indicated, so lines l and l' must intersect by **A3.** If $\varphi' < \phi$ lines l and l'

* Projective Geometry, Veblen and Young, vol. I, 1910, p. 16.

would not interesct, and if $\phi' = \phi$ the inverse function would not be single-valued. Thus $\phi' > \phi$, as was to be shown. Finally, **A1** requires that the function take on every value in the open interval $(0, \pi)$ for θ in that interval.

We have, then, that $f(\theta)$, for the closed interval $(0, \pi)$, is a single-valued strictly monotonic increasing function, with endpoints $(0, 0)$ and (π, π), taking on every value in the interval. Such a function is continuous in the closed interval, since the monotonicity requires that it can have simple discontinuities at most, in which event it could not take on all values in the interval.* The function $\phi = f(\theta)$ is thus of class S and the necessity of the condition is established.

We now assume that a geometry is defined by a function $\phi = f(\theta)$ of class S. Assumptions **A1** and **A2** will be satisfied if there is one and only one line through any two distinct points A and B. This is obvious when both A and B are in the same half-plane and when one or both points are on the axis. The remaining case is when A and B are in different half-planes, say A in the upper plane and B in the lower plane. The problem is to show that there is a unique breaking point along the axis such that the angles (θ, ϕ) formed by connecting the breaking point to A and B, respectively, satisfy the given functional relationship $\phi = f(\theta)$. It is evident geometrically that the function $\phi = g(\theta)$, obtained by varying the breaking point along the axis, is a single-valued strictly monotonic decreasing continuous function in the open interval $(0, \pi)$. This can be verified by dropping perpendiculars from A and B on the axis and noting that the function has the form $\cot \phi = a + b \cot \theta$, $b < 0$.† Defining $g(0) = \pi$ and $g(\pi) = 0$, we can describe $g(\theta)$ in the same terms used in defining $f(\theta)$, if we substitute "decreasing" for "increasing" and reverse the endpoint values. If the graphs of two functions so related were drawn on the same coordinate axes, the existence of the desired unique simultaneous solution would be geometrically evident. An analytic proof is obtained by consideration of the difference function $h(\theta) = g(\theta) - f(\theta)$. Since $h(\theta)$ is a single-valued strictly monotonic decreasing continuous function in the closed interval $(0, \pi)$ with endpoints $(0, \pi)$ and $(\pi, -\pi)$, it takes on the value zero once and only once, in the open interval $(0, \pi)$. Thus **A1** and **A2** are satisfied.

Assumption **A3** is satisfied if any two lines l and l', intersect. This is evident when at least one of the lines is the axis or a line parallel to the axis (ideal points exist as in euclidean geometry). When neither l nor l' is the axis or parallel to it, they both intersect the axis in ordinary points. If these two intersection points are not distinct this common point is the required intersection. For distinct intersections with the axis, refer to Figure 1, showing l and l'. If $\theta' < \theta$, there is an ordinary intersection in the upper plane, and if $\theta' = \theta$, there is an ideal intersection. If $\theta' > \theta$, we have $\phi' > \phi$ by the strict monotonicity condition on our function, and there is an ordinary intersection in the lower plane. In any event, an intersection exists and **A3** is satisfied. Thus the sufficiency of the condition is established, and we conclude that class S is the class of functions which define

* See Hardy, Pure Mathematics, 1925, p. 181.

† In this paper, the values of the inverse cotangent are not necessarily principal values, but are taken in the first and second quadrants.

the geometries satisfying the assumptions of alignment.

Note that the geometries defined by the functions of class S are actually plane geometries, since all points are on lines joining points of the axis to a particular point not on the axis.*

3. The non-desarguesian geometries. In what follows, we will refer interchangeably to a geometry and the function defining the geometry. All geometries considered will be understood to be of class S. Terminology involving an angle couple (θ, ϕ) of a geometry will refer to a pair of corresponding values of these angles in the open interval $(0, \pi)$ of the function defining the geometry. Note that two couples (θ, ϕ) and (θ', ϕ') of a geometry will necessarily have a strictly monotonic increasing relation to each other, *i.e.* $\phi' < \phi$ according as $\theta' \gtrless \theta$.

Consider two distinct couples of a geometry, (θ_1, ϕ_1) and (θ_2, ϕ_2), and draw the corresponding lines through a point P in the upper plane as shown in Figure 2. For any other couple (θ, ϕ) of the geometry, draw the line through P and construct triangles $A_1B_1C_1$ and $A_2B_2C_2$ as shown, with $A_1B_1 \| A_2B_2$ and $A_1C_1 \| A_2C_2$. If the geometry is assumed to be desarguesian, the two triangles, being perspective from the point P, would be perspective from the line at infinity. Then clearly, a second desarguesian geometry having the couples (θ_1, ϕ_1) and (θ_2, ϕ_2)

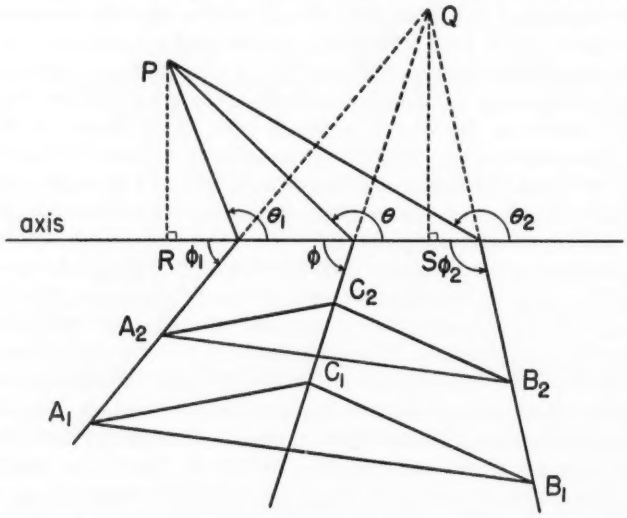

FIGURE 2

would have the couple (θ, ϕ) also, because C_1C_2P would be a line in the second geometry. Thus the two desarguesian geometries would be identical since (θ, ϕ)

* Reference is to Veblen and Young's conception of a plane and a plane geometry (*l.c.*, chap. I).

was any couple of the original desarguesian geometry other than the two common couples (θ_1, ϕ_1) and (θ_2, ϕ_2). We therefore arrive at

THEOREM 2. *With the possible exception of but one geometry, all the geometries of class S having two given distinct couples are non-desarguesian.*

COROLLARY. *All the geometries of class S having at least two couples in common with euclidean geometry (except euclidean geometry itself) are non-desarguesian.*

The corollary follows from the fact that euclidean geometry is desarguesian. As an application of the corollary, we note that Moulton's geometry is a non-desarguesian geometry of class S, since it is defined by

$$\begin{cases} \tan\phi = c\tan\theta, & \text{for} \quad 0 < \theta < \dfrac{\pi}{2},\ c > 0,\ c \neq 1 \\ \phi = \theta, & \text{for} \quad \theta = 0 \text{ and } \dfrac{\pi}{2} \leqq \theta \leqq \pi, \end{cases}$$

and has at least two couples in common with euclidean geometry (defined previously by $\phi = \theta,\ 0 \leqq \theta \leqq \pi$).

From the foregoing discussion of non-desarguesian geometries, it may be thought likely that all geometries of class S might be non-desarguesian (with the exception of euclidean geometry). This is not true, and the non-desarguesian proper subclass of S will now be determined.

Define T as the class of functions given by the two parameter family, $\cot\phi = a + b\cot\theta,\ b > 0,\ 0 < \theta < \pi$, with unique endpoints $(0, 0)$ and (π, π). We will prove

THEOREM 3. *The functions of class T define desarguesian geometries of class S.*

It is evident that a function of class T defines a geometry of class S. In order to show that Desargues' theorem holds for the geometries of class T, a general principle for these geometries will be established.* Consider a transformation taking any point P, at a height $p = PR$ above the axis, to a point Q, at a height $q = QS$ above the axis and at a distance $d = RS$ from P measured parallel to the axis with positive direction to the right (refer to Figure 2). Let the transformation equations be $q = (1/b)p$, and $d = (a/b)p$, where a and b are the parameters of a function of class T. If we take any couples of the geometry of this function of class T, such as (θ_1, ϕ_1), (θ_2, ϕ_2), and (θ, ϕ), it is seen that the euclidean prolongations of the lower plane segments of the corresponding lines through P will go through Q. We thus see that any configuration of points and lines in a geometry of class T can be transformed, by euclidean prolongations of the lower plane line segments, into a corresponding euclidean configuration such that concurrent lines go into concurrent lines and collinear points go into collinear points (the case of an ideal intersection is obvious, and the case of a line parallel to the axis is

* This principle was suggested by Professor S. B. Jackson.

handled by the first transformation equation). The converse situation is evidently true also. That is, upper plane lines and points of a euclidean configuration can be transformed back in the same way to yield a corresponding configuration in a geometry of class T. This establishes the general principle that a theorem relating to points of intersection is true in the geometries of class T if and only if it is true in euclidean geometry. In particular, then, we have Desargues' theorem true in the geometries of class T.

Noting that there is a geometry of class T having any two given distinct couples of a geometry of class S, we can remove the uncertainty in Theorem 2, and get

THEOREM 4. *With the exception of one desarguesian geometry (of class T), all the geometries of class S having two given distinct couples are non-desarguesian.*

Defining class $U \equiv S - T$, we have the theorem which may be regarded as the objective of this section.

THEOREM 5. *Class U is the class of non-desarguesian geometries of class S.*

4. A property of the non-desarguesian geometries. Pappus' theorem does not hold in our non-desarguesian projective planes by virtue of Hessenberg's proof that Desargues' theorem is a consequence of Pappus' theorem.* Hilbert and others have shown that the validity of Desargues' theorem in a projective plane is equivalent to the possibility of coordinatizing the plane by the von Staudt construction with elements from a skew-field, and that the skew-field will be a commutative field if and only if Pappus' theorem holds.†

The question arises as to what theorems can hold in our geometries of class U. We note that the method of proof used in the preceding section to show that these geometries are non-desarguesian can also be used to show that Pappus' theorem does not hold. The method can be used in general to show that all non-trivial constructible intersection point theorems of euclidean geometry are not true in our non-desarguesian planes. A definition of non-trivial constructible intersection point theorems is given by Moufang.‡ (The theorems of Desargues and Pappus and the theorem of the complete quadrilateral are examples of theorems of this type.)

To apply the method in general, consider a configuration for such a theorem of euclidean geometry constructed in a euclidean plane, with an axis drawn such that one of the intersection points of the configuration is isolated in the upper half-plane while the remaining intersection points are in the lower plane. An intersection point can be isolated by an axis in this manner since the definition

* G. Hessenberg, Beweis des Desarguesschen Satzes aus dem Pascalschen, Math. Ann., 61, 1905, pp. 161–172.

† See M. Pasch and M. Dehn (*l.c.*), and Marshall Hall, Projective planes, Trans. Amer. Math. Soc., 54, 1943, pp. 229–277.

‡ Ruth Moufang, Zur Struktur der projektiven Geometrie der Ebene, Math. Ann., 105, 1931, pp. 536–601.

of an intersection point theorem requires that there be only a finite number of intersection points in the configuration. Thus it is seen that an intersection point at the maximum distance from any particular point of the plane can be so isolated. Note that the definition of a non-trivial intersection point theorem also requires at least three lines on a point and three points on a line so that we have at least three lines of the configuration concurrent at the isolated point and cutting the axis in ordinary points. With the configuration of the theorem thus established in a non-desarguesian plane, it is seen that the arguments of the preceding section apply as before to show that the theorem is false in every one of the geometries of class U.

Thus far we have only considered non-trivial constructible intersection point theorems of euclidean geometry. However, we can show that possible theorems which do not hold in euclidean geometry cannot hold in our non-desarguesian planes either. For, consider the configuration of such a theorem constructed in a euclidean plane. Establish the configuration in a non-desarguesian plane by drawing an axis isolating it completely in the lower half-plane. Then, if the theorem were valid in the non-desarguesian geometry, it would hold in euclidean geometry also. We may now state

THEOREM 6. *All non-trivial constructible intersection point theorems are false in each of the geometries of class U.*

This yields, by elementary means, the result of Moufang (*l.c.*) to the effect that all non-trivial constructible intersection point theorems are independent of the plane axioms of projective geometry. It is interesting to note that a common geometric model (Moulton's geometry, for example) can be exhibited to show this independence for all non-trivial constructible intersection point theorems, and that each geometry of class U will serve as such a model.

CARDAN AND THE PASCAL TRIANGLE

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It is well known that the arithmetic triangle, in one or another variant of the form

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

appeared long before Pascal composed his famous *Traité du triangle arithmétique*

(published posthumously in 1665). The device may have been known to Omar Khayyam (c. 1100), it appeared in China in 1303, and it was published in Europe by Apian in 1529. [1] Similar forms of the triangle were given before Pascal by numerous men, including Stifel (1544), Scheubel (1545), Peletier (1549), Rudolph (1553), Tartaglia (1556), Stevin (1585), Girard (1629), Oughtred (1631), and Briggs (1633). [2] However, it seems still to be widely believed [3] that "the triangular array was investigated by Pascal (1654) under a new form, substantially as follows":

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots & & \\ 1 & 3 & 6 & \cdots & & & \\ 1 & 4 & \cdots & & & & \\ 1 & \cdots & & & & & \end{array}$$

So significant did Cantor, Wieleitner, and Tropfke regard the difference in form that they declined to recognize any dependence of Pascal upon the earlier instances. [4] There also is a general impression that to Pascal is due the first study of the relationships exhibited by the triangle and their application to questions in the theory of probability. [5] It is the purpose of this note to call attention to the work of one whose name has not been associated with the arithmetic triangle but who anticipated Pascal with respect both to the form and the study of the triangle. This man, perhaps the greatest mathematician of his day, was Jerome Cardan (1501–1576).

The *Ars magna*, which appeared in 1545, contains no reference to the arithmetic triangle; but in 1570 Cardan published his *Opus novum de proportionibus*, and in this work the Pascal triangle appears in both forms and with varying applications. In connection with the problem of the determination of roots of numbers, Cardan used the familiar earlier form, citing Stifel as the putative discoverer. Here he gave the numbers in the triangle through $n=17$, and he pointed out the relationship, known to Stifel, equivalent to

$$\binom{m}{n}+\binom{m}{n+1}=\binom{m+1}{n+1}$$

[6]. Later on in the *Opus novum* the question of combinations and probabilities is taken up, and then Cardan gave the arithmetic triangle, through $n=11$, in virtually the form later made famous by Pascal. [7] Even in the case of Cardan, however, this form was not entirely original, for in 1556 Tartaglia had published it in a somewhat similar square array, [8] and had used it in determining the coefficients in the expansion of the twelfth power (*cubo-censo-censo*) of a binomial (*una quantita divisia in due parti*). In connection with the newer arrangement of the triangle Cardan reiterated the familiar formula known by his name and enunciated by him many years before—the total number of different combina-

tions which can be formed from n different objects, taking as many at a time as one pleases, is given by 2^n-1. As an illustration of the usefulness of his result, he determined the number of possible combinations of 25 different objects, the result being $2^{25}-1=16777215$. Following this, Cardan gave something more important which seems to have been overlooked by historians and which incorrectly has been ascribed to Pascal. [9] This is the rule of succession, equivalent to

$$\binom{n}{r}=\frac{n-r+1}{r}\binom{n}{r-1}.$$

The rule is not given symbolically but is expressed rather awkwardly as follows: To obtain the third element in the row corresponding to $n=11$, for example, subtract one from eleven, divide by two, and then multiply by eleven. The result, 55, is the number desired. To obtain the next number, subtract two from eleven, divide by three, and multiply by 55. Continuing in this manner, Cardan obtained all of the elements in the sequence of combinations of eleven objects. [10] Had Cardan applied his rule to the expansion of binomials, he would have anticipated the binomial theorem for positive integral powers. Instead he emphasized the connection between the numbers of the arithmetic triangle and "mixed proportions," *i.e.*, progressions of higher order, and the applicability of these to musical theory. There can be little doubt but that Cardan, like Tartaglia, was aware that the elements in the triangle are coefficients in the expansions of binomials. That a clear-cut statement of Cardan's rule of succession as applied to the binomial theorem should have been delayed for about another century is one of the anomalies in the development of mathematics, and that the arithmetic triangle should be named for Pascal rather than for one of his many anticipators is largely an accident of history.

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6. Jerome Cardan, Opera omnia (10 vols., Lugduni, 1663), IV, 529.

7. *Ibid.*, IV, 557.

8. Niccolo Tartaglia, General trattato di numeri et misure (part II, Vinegia, 1556), book I, fol. 17; book II, fols. 69–73. H. G. Zeuthen, Geschichte der Mathematik im XVI. und XVII. Jahrhundert (German ed., Leipzig, 1903), pp. 102, 169, and Guillaume Libri, Histoire des sciences

mathématiques en Italie (4 vols., Paris, 1838–1841), III, 158, mention the work of Tartaglia but not that of Cardan.

9. Encyclopédie des sciences mathématiques, I (1), 1 (1904), pp. 83–84. Tropfke, *loc. cit.*, ascribes Cardan's rule of succession to Briggs in 1633.

10. Opera omnia, IV, 558.

THE FINITE FOURIER SERIES AND ELEMENTARY GEOMETRY

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Every freshman knows, or soon learns, that the periodic sequence $x_0=1$ $x_1=-1$, $x_2=1$, $x_3=-1$, . . . has a general term x_n which may be written as $x_n=(-1)^n$. This is a very special case of representation of a periodic sequence by the so-called finite Fourier Series. The finite Fourier series is the analogue for periodic sequences of the ordinary Fourier series expansion of periodic functions. Its only mention in our textbook literature is under the heading of practical harmonic analysis, or trigonometric interpolation, as found in books on applied mathematics or numerical methods; and yet, the range of applications of the finite Fourier series extends beyond this important practical problem. It is intimately related to the Gaussian sums and has been used for number theoretic purposes by Eisenstein and more recently by H. A. Rademacher. The following lines present the basic properties of the (complex) finite Fourier series stressing its geometric significance and followed by a few applications to extremal problems of elementary geometry.

§1. The Finite Fourier Series

1. The finite Fourier series. We recall the general problem of polynomial interpolation: If we are given k distinct complex numbers $\omega_0, \omega_1, \cdots, \omega_{k-1}$, and a second set of k arbitrary complex numbers $z_0, z_1, \cdots, z_{k-1}$, then there is one and only one polynomial $P(x)=\zeta_0+\zeta_1 x+\cdots+\zeta_{k-1}x^{k-1}$ satisfying the equations $P(\omega_\nu)=z_\nu$ $(\nu=0, 1, \cdots, k-1)$ or

$$(1) \qquad z_\nu = \zeta_0 + \zeta_1\omega_\nu + \zeta_2\omega_\nu^2 + \cdots + \zeta_{k-1}\omega_\nu^{k-1} \qquad (\nu = 0, 1, \cdots, k-1). \text{ [1]}$$

Indeed, the system of linear equations (1) in the unknowns ζ_ν has a non-vanishing (Vandermonde) determinant and has therefore a unique solution. We obtain the finite Fourier series (abbreviated in the sequel to f.F.S.) if we choose the numbers ω_ν to be the kth roots of unity; so for the remainder of this paper we shall assume that

$$(2) \qquad \omega_\nu = e^{2\pi i\nu/k} \qquad (\nu = 0, 1, \cdots, k-1).$$

This equation defines ω_ν for all integral values of ν, a fact occasionally used later.

The equations (1) now represent, by definition, the f.F.S. expansion of the given sequence $z_0, \cdots, z_{k-1}$, while the coefficients $\zeta_0, \cdots, \zeta_{k-1}$, are called the f.F. coefficients of the given sequence (z_ν).

In order to solve the system (1), *i.e.* find the f.F. coefficients ζ_ν, a few properties of the roots of unity (2) are necessary. Thus (2) readily implies such relations as

$$\omega_\nu^\alpha = \omega_\alpha^\nu, \qquad \bar{\omega}_\nu^\alpha = \omega_\nu^{-\alpha}, \qquad \omega_1^\alpha = \omega_\alpha, \qquad \omega_{\alpha+1} = \omega_\alpha \omega_1, \tag{3}$$

which will be used freely. Especially important for us are the so-called orthogonality relations

$$\sum_{\nu=0}^{k-1} \omega_\nu^\alpha \bar{\omega}_\nu^\beta = \begin{cases} k & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta, \end{cases} \qquad (\alpha, \beta = 0, 1, \cdots, k-1). \tag{4}$$

They allow us to determine the f.F. coefficients as follows: If we multiply (1) or

$$z_\nu = \sum_{\mu=0}^{k-1} \zeta_\mu \omega_\nu^\mu$$

by $\bar{\omega}_\alpha^\nu = \bar{\omega}_\nu^\alpha$ and sum over all ν, we find

$$\sum_{\nu=0}^{k-1} z_\nu \bar{\omega}_\alpha^\nu = \sum_{\mu,\nu} \zeta_\mu \omega_\nu^\mu \bar{\omega}_\nu^\alpha = \sum_{\mu=0}^{k-1} \zeta_\mu \sum_{\nu=0}^{k-1} \omega_\nu^\mu \bar{\omega}_\nu^\alpha = \zeta_\alpha \cdot k$$

by (4). The f.F. coefficients are therefore given by the relations

$$\zeta_\alpha = \frac{1}{k}\left(z_0 + z_1 \bar{\omega}_\alpha + z_2 \bar{\omega}_\alpha^2 + \cdots + z_{k-1} \bar{\omega}_\alpha^{k-1}\right), \quad (\alpha = 0, 1, \cdots, k-1). \tag{5}$$

2. Geometric interpretation. It is both suggestive and desirable for future purposes to plot the numbers z_ν in the complex plane where they may be thought of as the successive vertices of a closed polygon $\Pi = (z_\nu)$. In particular, the sequence (ω_ν) represents there a regular k-gon $\Pi_1 = (\omega_\nu)$, while the sequence (ω_ν^α) represents a k-gon $\Pi_\alpha = (\omega_\nu^\alpha)$ obtained from Π_1 by starting from $\omega_0 = 1$ and taking every αth vertex. Π_α is a starred regular k-gon if and only if α is relatively prime to k; otherwise it is a regular polygon of lower order described several times so as to result in a total of k vertices. We also include the polygon $\Pi_0 = (\omega_\nu^0)$ all of whose k vertices coincide with $\omega_0 = 1$. A glance at the f.F.S. (1) now shows that our polygon $\Pi = (z_\nu)$ has been analyzed harmonically, *i.e.* has been represented as a linear combination of the fundamental k-gons Π_α in a manner suggested by the symbolic relation $\Pi = \zeta_0\Pi_0 + \zeta_1\Pi_1 + \cdots + \zeta_{k-1}\Pi_{k-1}$.

It is sometimes desirable to re-write the expansion (1) by grouping its terms as follows:

$$z_\nu = \zeta_0 + (\zeta_1 \omega_\nu + \zeta_{k-1} \omega_\nu^{k-1}) + (\zeta_2 \omega_\nu^2 + \zeta_{k-2} \omega_\nu^{k-2}) + \cdots,$$

or

$$z_\nu = \zeta_0 + (\zeta_1\omega_\nu + \zeta_{k-1}\bar{\omega}_\nu) + (\zeta_2\omega_\nu^2 + \zeta_{k-2}\bar{\omega}_\nu^2) + \cdots, \tag{6}$$

where the last term is the central term of (1), or the sum of the two central terms of (1), depending on whether k is even or odd. Also the individual terms of (6) have a simple geometric interpretation which I learned orally from Fritz John. Let us look at the first term

$$z'_\nu = \zeta_1\omega_\nu + \zeta_{k-1}\bar{\omega}_\nu, \qquad (\nu = 0, \cdots, k-1). \tag{7}$$

Separating reals and imaginaries by setting

$$z'_\nu = x'_\nu + iy'_\nu, \quad \zeta_1 = a_1 + ib_1, \quad \zeta_{k-1} = c_1 + id_1, \quad \omega_\nu = \cos\frac{2\pi\nu}{k} + i\sin\frac{2\pi\nu}{k},$$

we find the coordinates of the point (7) to be

$$x'_\nu = (a_1 + c_1)\cos\frac{2\pi\nu}{k} + (-b_1 + d_1)\sin\frac{2\pi\nu}{k},$$

$$y'_\nu = (b_1 + d_1)\cos\frac{2\pi\nu}{k} + (a_1 - c_1)\sin\frac{2\pi\nu}{k}.$$

If we write

$$a' = a_1 + c_1, \qquad b' = -b_1 + d_1,$$
$$c' = b_1 + d_1, \qquad d' = a_1 - c_1,$$

it is clear that the polygon Π'_1, of vertices (7), is the image of the regular k-gon $\Pi_1 = (\omega_\nu)$ by the affine transformation

$$\begin{aligned} x' &= a'x + b'y, \\ y' &= c'x + d'y. \end{aligned} \tag{8}$$

Conversely, by following our relations backwards we see that we may start with an arbitrary affine transformation (8) and find that the image of Π_1, by (8), may be written in the form (7). Generally, if α is any integer in the range $1 \leqq \alpha \leqq k/2$, we find the αth term of (6), or

$$z_\nu^{(\alpha)} = \zeta_\alpha\omega_\nu^\alpha + \zeta_{k-\alpha}\bar{\omega}_\nu^\alpha \qquad (\nu = 0, \cdots, k-1), \tag{9}$$

to represent the image $\Pi_\alpha^{(\alpha)}$, of $\Pi_\alpha = (\omega_\nu^\alpha)$, by a uniquely defined affine transformation

$$\begin{aligned} x^{(\alpha)} &= a^{(\alpha)}x + b^{(\alpha)}y, \\ y^{(\alpha)} &= c^{(\alpha)}x + d^{(\alpha)}y, \end{aligned} \tag{10}$$

whose coefficients are related to those of (9) by precisely the same relations which connected (8) to (7).

We may state our result as follows: *The expansion* (6) *represents our given polygon* Π *as a uniquely defined sum of a constant* ζ_0 (*the centroid of* Π) *plus* $[k/2]$ *k-gons* $\Pi_\alpha^{(\alpha)}$ $(\alpha=1, \cdots, [k/2])$, *where* $\Pi_\alpha^{(\alpha)}$ *is an affine image of the fundamental k-gon* $\Pi_\alpha=(\omega_\nu^\alpha)$. [2]

The expansion (6) should be called the real f.F.S. of (z_ν), in contradistinction to (1) which is the complex f.F.S. These terms are justified by the following remarks:

1. If all z_ν are real then ζ_0 is real and $\zeta_{k-\alpha}=\bar{\zeta}_\alpha$ $(\alpha=1, 2, \cdots, k-1)$, and conversely. Indeed, the direct statement follows from (5), the converse one from (6).

2. If the z_ν are real then all terms of the real f.F.S. (6) are real. Evidently so, because $\zeta_\alpha\omega_\nu^\alpha+\zeta_{k-\alpha}\bar{\omega}_\nu^\alpha$ is the sum of two complex conjugate terms. The same result can be seen from (9) which turns out to be a degenerate affine transformation with $y^{(\alpha)}=0$, *i.e.* mapping the whole plane onto the real axis.

3. The analogue of the Parseval relation. [3] Besides the f.F.S.

$$z_\nu = \sum_{\alpha=0}^{k-1} \zeta_\alpha \omega_\nu^\alpha, \qquad (\nu = 0, \cdots, k-1), \tag{11}$$

let

$$x_\nu = \sum_{\beta=0}^{k-1} \xi_\beta \omega_\nu^\beta, \qquad (\nu = 0, \cdots, k-1), \tag{12}$$

be the f.F.S. of a second sequence (x_ν). We now find

$$\sum_{\nu=0}^{k-1} z_\nu \bar{x}_\nu = \sum_{\alpha,\beta,\nu} \zeta_\alpha \omega_\nu^\alpha \bar{\xi}_\beta \bar{\omega}_\nu^\beta = \sum_{\alpha,\beta} \zeta_\alpha \bar{\xi}_\beta \sum_\nu \omega_\nu^\alpha \bar{\omega}_\nu^\beta$$

and therefore, by the orthogonality relations (4),

$$\sum_{\nu=0}^{k-1} z_\nu \bar{x}_\nu = k \cdot \sum_{\alpha=0}^{k-1} \zeta_\alpha \bar{\xi}_\alpha.$$

In particular, if $x_\nu=z_\nu$ we find that

$$\sum_{\nu=0}^{k-1} |z_\nu|^2 = k \cdot \sum_{\alpha=0}^{k-1} |\zeta_\alpha|^2. \tag{13}$$

This is the finite Parseval relation which we shall use repeatedly.

4. The convolution of sequences. Let again (z_ν) and (x_ν) be the two sequences (11) and (12). Out of them we form a new sequence (z_ν') defined by the equations

$$z_\nu' = \sum_{\alpha=0}^{k-1} z_\alpha x_{\nu-\alpha} \qquad (\nu = 0, \cdots, k-1), \tag{14}$$

where we observe the convention of treating (x_ν) as an infinite sequence of period k, *i.e.* $x_{-1}=x_{k-1}$, $x_{-2}=x_{k-2}$, and so forth. In explicit form, the relations (14) are

$$
\begin{aligned}
z_0' &= z_0x_0 + z_1x_{k-1} + z_2x_{k-2} + \cdots + z_{k-1}x_1 \\
z_1' &= z_0x_1 + z_1x_0 + z_2x_{k-1} + \cdots + z_{k-1}x_2 \\
&\cdots\cdots\cdots\cdots \\
z_{k-1}' &= z_0x_{k-1} + z_1x_{k-2} + z_2x_{k-3} + \cdots + z_{k-1}x_0.
\end{aligned} \tag{15}
$$

We say that (z_ν') is obtained from the sequences (z_ν) and (x_ν) by convolution, an operation which we indicate by the symbolic relation $(z_\nu')=(z)_\nu*(x_\nu)$. The f.F.S. of (z_ν') is readily found as follows: From (14), (11) and (12) we have

$$
\begin{aligned}
z_\nu' &= \sum_\alpha z_\alpha x_{\nu-\alpha} = \sum_{\lambda,\mu,\alpha} \zeta_\lambda \omega_\alpha^\lambda \xi_\mu \omega_{\nu-\alpha}^\mu = \sum_{\lambda,\mu,\alpha} \zeta_\lambda \xi_\mu \omega_\nu^\mu \omega_\alpha^\lambda \bar{\omega}_\alpha^\mu \\
&= \sum_{\lambda,\mu} \zeta_\lambda \xi_\mu \omega_\nu^\mu \sum_{\alpha=0}^{k-1} \omega_\alpha^{\lambda-\mu},
\end{aligned}
$$

and finally, by (4),

$$
z_\nu' = \sum_{\lambda=0}^{k-1} (\zeta_\lambda \xi_\lambda k)\omega_\nu^\lambda, \qquad (\nu = 0, \cdots, k-1).
$$

This being the f.F.S. of (z_ν'), we see that the f.F. coefficients (ζ_ν') of the convolution $(z_\nu')=(z_\nu)*(x_\nu)$ are given by

$$
\zeta_\nu' = \zeta_\nu \xi_\nu k \qquad (\nu = 0, \cdots, k-1). \tag{16}
$$

We wish to state this result in a somewhat more convenient form. We may look on the convolution operation $(z_\nu')=(z_\nu)*(x_\nu)$ as the linear transformation (15) from the variables (z_ν) to the new variables (z_ν'). (15) may be described as a cyclic transformation because the successive rows of its matrix are obtained from its first row by successive cyclic permutations. Now (16) shows that the f.F. coefficients of (z_ν') are obtained by multiplying those of (z_ν), *i.e.* the (ζ_ν), by the factors

$$
\xi_\nu k = x_0 + x_1\bar{\omega}_\nu + x_2\bar{\omega}_\nu^2 + \cdots + x_{k-1}\bar{\omega}_\nu^{k-1}
$$

which we prefer to write in the equivalent form

$$
\xi_\nu k = x_0 + x_{k-1}\omega_\nu + x_{k-2}\omega_\nu^2 + \cdots + x_1\omega_\nu^{k-1}.
$$

If we change our notation by defining $a_0=x_0$, $a_1=x_{k-1}$, $a_2=x_{k-2}$, $\cdots$, $a_{k-1}=x_1$, we may state our result as

THEOREM 1. *If (ζ_ν) are the f.F. coefficients of a sequence (z_ν), and if we subject the sequence (z_ν) to the cyclic transformation*

$$
\begin{aligned}
z_0' &= a_0z_0 + a_1z_1 + a_2z_2 + \cdots + a_{k-1}z_{k-1} \\
z_1' &= a_{k-1}z_0 + a_0z_1 + a_1z_2 + \cdots + a_{k-2}z_{k-1} \\
&\vdots \\
z_{k-1}' &= a_1z_0 + a_2z_1 + a_3z_2 + \cdots + a_0z_{k-1}
\end{aligned} \tag{17}
$$

then the f.F. coefficients of the new sequence (z_ν') are

$$\zeta_\nu' = \zeta_\nu f(\omega_\nu) \qquad (\nu = 0, \cdots, k-1), \tag{18}$$

where

$$f(z) = a_0 + a_1z + \cdots + a_{k-1}z^{k-1}, \tag{19}$$

and will be referred to as the representative polynomial of the cyclic transformation (17). [4]

By the finite Parseval relation (13) we have

$$\sum |z_\nu'|^2 = k \sum_\nu |\zeta_\nu'|^2.$$

If we substitute on the right-hand side the values (18) we obtain the following

COROLLARY. *The relations* (17) *imply the identity*

$$\sum_{\nu=0}^{k-1} |z_\nu'|^2 = k \cdot \sum_{\nu=0}^{k-1} |f(\omega_\nu)|^2 |\zeta_\nu|^2, \tag{20}$$

where (ζ_ν) are the f.F. coefficients of the sequence (z_ν), while $f(z)$ is defined by (19).

The relation (20) should be regarded as an identity in the k variables (z_ν), the (ζ_ν) being given in terms of the (z_ν) by (5). It has a number of applications to elementary geometry to which we now turn.

§2. GEOMETRIC APPLICATIONS

5. The MONTHLY Problem no. 3547. The problem referred to was proposed by Martin Rosenman in 1932 and may be stated as follows:

Let $\Pi = P_0P_1 \cdots P_{k-1}$ be a closed polygon in the plane $(k \geqq 2)$. Denote by $P_0', P_1', \cdots, P_{k-1}$, the midpoints of the sides $P_0P_1, P_1P_2, \cdots, P_{k-1}P_0$, respectively, obtaining the first derived polygon $\Pi' = P_0'P_1' \cdots P_{k-1}'$. Repeat the same construction on Π' obtaining the second derived polygon Π'', and finally, after n constructions, obtain the nth derived polygon $\Pi^{(n)} = P_0^{(n)}P_1^{(n)} \cdots P_{k-1}^{(n)}$. Show that the vertices of $\Pi^{(n)}$ converge, as $n \to \infty$, to the centroid of the original points $P_0, P_1, \cdots, P_{k-1}$. [5].

Let $z_0, \cdots, z_{k-1}$ denote the complex coordinates of the vertices of Π. It is then clear that the coordinates of the vertices of the polygon Π' are obtained from (z_ν) by means of a linear cyclic transformation (17) of coefficients $a_0=a_1=1/2$, $a_2=\cdots=a_{k-1}=0$. Its representative polynomial (19) now becomes

$$f(z) = (1+z)/2,$$

and by (20) we have the identity

$$\sum_{\nu=0}^{k-1} |z'_\nu|^2 = k \cdot \sum_{\nu=0}^{k-1} \left|\frac{1+\omega_\nu}{2}\right|^2 \cdot |\zeta_\nu|^2, \tag{21}$$

where (ζ_ν) are the f.F. coefficients of (z_ν). We now make the further assumption

$$\sum_{0}^{k-1} z_\nu = 0, \tag{22}$$

which means that the centroid of the vertices of Π is at the origin. By (5), our assumption (22) is equivalent to $\zeta_0=0$, so that (21) now implies

$$\begin{aligned}\sum_{0}^{k-1} |z'_\nu|^2 &= k \cdot \sum_{\nu=1}^{k-1} \left|\frac{1+\omega_\nu}{2}\right|^2 \cdot |\zeta_\nu|^2 \leqq \max_{\nu=1,\cdots,k-1} \left|\frac{1+\omega_\nu}{2}\right|^2 \cdot k \sum_{\nu=1}^{k-1} |\zeta_\nu|^2 \\ &= \cos^2\frac{\pi}{k} \cdot \sum_{0}^{k-1} |z_\nu|^2.\end{aligned} \tag{23}$$

In justifying the last equality sign of (23) we have used firstly, the finite Parseval relation (13) with $\zeta_0=0$, secondly, that the maximum occurring in (23) is $\cos^2(\pi/k)$. This last fact is immediately seen geometrically, because the points $(1+\omega_\nu)/2$, which are the midpoints of the segments joining $z=1$ to $z=\omega_\nu$, are the vertices of a regular polygon inscribed in the circle having the segment joining $z=0$ to $z=1$ as diameter. This proves the following

THEOREM 2. *Let $z_0, z_1, \cdots, z_{k-1}$ be the vertices of a k-gon such that $\sum z_\nu=0$, and let $z'_0, z'_1, \cdots, z'_{k-1}$ be the midpoints of its sides, then*

$$\sum_{\nu=0}^{k-1} |z'_\nu|^2 \leqq \cos^2\frac{\pi}{k} \cdot \sum_{\nu=0}^{k-1} |z_\nu|^2, \tag{24}$$

with equality if and only if

$$z_\nu = \zeta_1\omega_\nu + \zeta_{k-1}\bar{\omega}_\nu, \qquad (\nu = 0, \cdots, k-1), \tag{25}$$

i.e. when our polygon Π is an affine transform of an ordinary regular k-gon with its center at the origin.

The inequality (24) follows from (23). To find the cases when we have the equality sign in (24) we need the further geometrically obvious fact that the equality

$$\left|\frac{1+\omega_\nu}{2}\right|^2 = \max_{\nu=1,\cdots,k-1}\left|\frac{1+\omega_\nu}{2}\right|^2 = \cos^2\frac{\pi}{k}$$

is true only if $\nu=1$, or $\nu=k-1$. Now we have equality in (24), *i.e.* in (23), if and only if $\zeta_\nu=0$ for $\nu=2, 3, \cdots, k-2$, which means that the f.F.S. of (z_ν) indeed reduces to (25).

As special cases we mention the following: 1. If $k=3$ we always have equality in (24). 2. If $k=4$ we have equality in (24) if and only if the quadrilateral Π is a parallelogram.

A solution of Problem no. 3547 is now immediate; indeed, it is clear from (18) for $\nu=0$ (or also directly) that we have $\zeta_0'=0$, hence $\sum z_\nu'=0$. We may therefore apply the inequality (24) to any pair of consecutive derived polygons $\Pi^{(s-1)}$, $\Pi^{(s)}$, obtaining

$$\sum_{\nu=0}^{k-1}\left|z_\nu^{(s)}\right|^2 = \cos^2\frac{\pi}{k}\cdot\sum_{\nu=0}^{k-1}\left|z_\nu^{(s-1)}\right|^2.$$

From these equations for $s=1, 2, \cdots, n$ we obtain

$$\sum_{\nu=0}^{k-1}\left|z_\nu^{(n)}\right|^2 \leqq \cos^{2n}\frac{\pi}{k}\cdot\sum_{\nu=0}^{k-1}\left|z_\nu\right|^2. \tag{26}$$

As $n\to\infty$, the right-hand side tends to zero, because $\cos(\pi/k)<1$, showing that

$$\lim_{n\to\infty} z_\nu^{(n)} = 0, \qquad (\nu = 0, \cdots, k-1),$$

as required by the problem.

Our last inequality (26) gives some idea of the rapidity of convergence and of its deterioration as k increases. We actually have equality in (26) for all n if and only if Π is an affine image of a regular polygon. More precise information concerning the sequence of polygons $\Pi^{(n)}$ is furnished by (18). For instance if $k=4$, then $f(\omega_2)=(1+\omega_2)/2=0$, therefore Π' and all succeeding $\Pi^{(n)}$ are parallelograms, a fact which is evident geometrically.

6. A generalization of Problem no. 3547. Our generalization is as follows. We start as before from a polygon $\Pi=(z_\nu)$ and derive from it the new polygon $\Pi'=(z_\nu')$, not by the special midpoint construction as heretofore, but by means of a fixed cyclic transformation (17). Let again $\Pi^{(n)}$ be the nth polygon derived by repeating the construction (17) n times. Under what circumstances will the vertices $P_0^{(n)}, \cdots, P_{k-1}^{(n)}$ of the polygon $\Pi^{(n)}$ converge to limit positions for an arbitrary choice of the original polygon Π? [6]

This question is readily settled if we employ the finite Fourier series of our successive polygons and Theorem 1. Indeed, if $(\zeta_\nu^{(n)})$ are the f.F. coefficients of $\Pi^{(n)}$ $(n=0, 1, \cdots)$, we find by a repeated application of (18) that

$$\zeta_\nu^{(n)} = (f(\omega_\nu))^n\zeta_\nu, \qquad (\nu = 0, \cdots, k-1). \tag{27}$$

Therefore the $\lim \zeta_\nu^{(n)}$, as $n \to \infty$, will exist for all ν and arbitrary values of (ζ_ν) if and only if, for each ν, we have either $f(\omega_\nu) = 1$, or else $|f(\omega_\nu)| < 1$. We may state our result as

THEOREM 3. *Let $\Pi = (z_\nu)$ be a plane k-gon and let $\Pi' = (z'_\nu)$ be the polygon derived from Π by the construction described by the cyclic linear transformation* (17). *Let $\Pi'' = (z'')$ be the polygon obtained from Π' by the same construction* (17) *and finally, let $\Pi^{(n)} = (z_\nu^{(n)})$ be the polygon obtained after n iterations of the construction* (17). *The k limits*

$$\lim_{n \to \infty} z_\nu^{(n)} = z_\nu^{(\infty)}, \qquad (\nu = 0, \cdots, k-1), \tag{28}$$

will exist for every polygon Π if and only if all of the k roots of unity (ω_ν) fall into two classes (ω_ρ) and (ω_σ) such that

$$\begin{aligned} f(\omega_\rho) &= 1 \quad \textit{for the roots of the first class } (\omega_\rho), \\ |f(\omega_\sigma)| &< 1 \quad \textit{for the roots of the second class } (\omega_\sigma). \end{aligned} \tag{29}$$

If these conditions are satisfied, then the vertices of the limit polygon are given by

$$z_\nu^{(\infty)} = \sum_{(\rho)} \zeta_\rho \omega_\nu^\rho \qquad (\nu = 0, \cdots, k-1), \tag{30}$$

which is the sum of precisely those terms of the f.F.S. (1) *of the original polygon $\Pi = (z_\nu)$ which correspond to the first class of subscripts (ρ) according to* (29). *In particular, the limit of $\Pi^{(n)}$ is the centroid of the vertices of Π if and only if the f.F.S.* (30) *reduces to the constant term ζ_0 of* (1), *which is the case if and only if*

$$f(1) = 1, \qquad |f(\omega_\nu)| < 1 \quad \textit{if} \quad \nu = 1, 2, \cdots, k-1. \tag{31}$$

This theorem has already been established up to the conditions (29) inclusive. If they are verified we get, letting $n \to \infty$ in (27), the values

$$\zeta_\rho^{(\infty)} = \zeta_\rho, \qquad \zeta_\sigma^{(\infty)} = 0,$$

for the f.F. coefficients of the limit polygon $\Pi^{(\infty)}$. But then the f.F.S. of $\Pi^{(\infty)}$ is indeed (30).

7. An extremal problem. The argument which led from the identity (20) to Theorem 2 may be readily extended so as to furnish a general inequality, of the same type as (24), in terms of a general cyclic transformation (17). We prefer, however, to discuss only one further special case. Starting as before from the polygon $\Pi = (z_\nu)$, we consider the special cyclic transformation

$$z'_\nu = z_{\nu+1} - z_\nu \qquad (\nu = 0, \cdots, k-1; z_k = z_0). \tag{32}$$

This is indeed identical with (17) if $a_0 = -1$, $a_1 = 1$, $a_2 = \cdots = a_{k-1} = 0$, hence

$$f(z) = -1 + z.$$

But then (20) reduces to the identity

$$\sum_{\nu} |z_{\nu+1} - z_\nu|^2 = k \cdot \sum_{\nu} |\omega_\nu - 1|^2 \cdot |\zeta_\nu|^2. \tag{33}$$

From this we draw two conclusions: Firstly

$$\begin{aligned} \sum_{\nu} |z_{\nu+1} - z_\nu|^2 &= k \sum_{\nu} |\omega_\nu - 1|^2 \cdot |\zeta_\nu|^2 \\ &\leqq \left(\max_{\nu} |\omega_\nu - 1|^2\right) \cdot k \sum_{\nu} |\zeta_\nu|^2 \\ &= \left(\max_{\nu} |\omega_\nu - 1|^2\right) \cdot \sum_{\nu} |z_\nu|^2, \end{aligned} \tag{34}$$

by the finite Parseval relation. Secondly, assuming that $\sum z_\nu = 0$, or $\zeta_0 = 0$, we have

$$\begin{aligned} \sum_{\nu} |z_{\nu+1} - z_\nu|^2 &= k \cdot \sum_{\nu=1}^{k-1} |\omega_\nu - 1|^2 \cdot |\zeta_\nu|^2 \\ &\geqq \left(\min_{\nu=1,\cdots,k-1} |\omega_\nu - 1|^2\right) \cdot k \sum_{\nu=1}^{k-1} |\zeta_\nu|^2 \\ &= \left(\min_{\nu=1,\cdots,k-1} |\omega_\nu - 1|^2\right) \cdot \sum_{0}^{k-1} |z_\nu|^2. \end{aligned} \tag{35}$$

This proves

THEOREM 4. *Let $\Pi = P_0P_1 \cdots P_{k-1}$ be a plane closed k-gon having the point 0 as centroid of its vertices. The following inequalities then hold*

$$4 \sin^2 \frac{\pi}{k} \leqq \frac{\sum_{\nu} (P_\nu P_{\nu+1})^2}{\sum_{\nu} (OP_\nu)^2} \leqq 4 \sin^2 \left(\frac{\pi}{k} \cdot \left[\frac{k}{2}\right]\right), \tag{36}$$

where we have equality on the left side if and only if Π is an affine image of an ordinary regular k-gon; the equality sign holds on the right side under circumstances which depend on the parity of k as follows:

1. *If $k = 2p$, we have equality on the right if and only if*

$$P_0 = P_2 = \cdots = P_{2p-2} \quad \text{and} \quad P_1 = P_3 = \cdots = P_{2p-1},$$

i.e. our polygon is a segment described to and fro p times. The right-hand side of (36) *reduces to the value* 4.

2. *If $k = 2p+1$, we have equality on the right if and only if Π is an affine image of that starred regular k-gon, inscribed in the unit circle, which has the largest side among all such starred regular k-gons. The right-hand side of* (36) *reduces to the value* $4 \cos^2 (\pi/2k)$.

The conditions of equality in (36) follow by inspection of the multiple relations (34) and (35). Thus the extreme right member of (36) is the value of the maximum occurring in (34). Equality occurs on the right in (36) if and only if we have equality in (34). If $k=2p$, this is the case if and only if $\zeta_0=\cdots=\zeta_{p-1}=\zeta_{p+1}=\cdots=\zeta_{k-1}=0$, *i.e.* the f.F.S. of Π reduces to

$$z_\nu=\zeta_p\omega_\nu^p=\zeta_p(-1)^\nu.$$

If $k=2p+1$ we have equality in (34) if and only if the f.F.S. of Π reduces to

$$z_\nu=\zeta_p\omega_\nu^p+\zeta_{p+1}\omega_\nu^{p+1}=\zeta_p\omega_\nu^p+\zeta_{p+1}\bar{\omega}_\nu^p,$$

which is an affine image of the starred polygon $\Pi_p=(\omega_\nu^p)$ described in the theorem (compare with (9)). The conditions of equality on the left side of (36) are obtained by a similar argument involving (35).

A few special cases are worth mentioning separately:

If $k=3$, the extreme members of (36) are equal and are 3, whence the identity

$$\sum_{\nu=0}^{2}(P_\nu P_{\nu+1})^2=3\cdot\sum_{\nu=0}^{2}(OP_\nu)^2.$$

If $k=4$ we have

$$2\leqq\frac{\sum_0^3(P_\nu P_{\nu+1})^2}{\sum_0^3(OP_\nu)^2}\leqq 4,$$

with equality on the left if and only if $P_0P_1P_2P_3$ is a parallelogram, and equality on the right if and only if $P_0=P_2$, $P_1=P_3$.

8. The area of polygons. Let A denote the oriented area of the polygon $\Pi=(z_0, z_1, \cdots, z_{k-1})$. Let $z_\nu=x_\nu+iy_\nu$; then

$$\text{Area of triangle } Oz_\nu z_{\nu+1}=\tfrac{1}{2}(x_\nu y_{\nu+1}-x_{\nu+1}y_\nu)=\tfrac{1}{2}\Im(\bar{z}_\nu z_{\nu+1}).$$

The sum of the areas of these k triangles being the area of the k-gon, we have

$$A=\frac{1}{2}\sum_{\nu=0}^{k-1}\Im(\bar{z}_\nu z_{\nu+1})=\frac{1}{4i}\sum_{\nu=0}^{k-1}(\bar{z}_\nu z_{\nu+1}-z_\nu\bar{z}_{\nu+1}). \tag{37}$$

The area A may now readily be expressed in terms of the f.F. coefficients of Π. Indeed, by (1) we have

$$\sum_\nu\bar{z}_\nu z_{\nu+1}=\sum_{\alpha,\beta,\nu}\bar{\zeta}_\alpha\bar{\omega}_\nu^\alpha\zeta_\beta\omega_{\nu+1}^\beta=\sum_{\alpha,\beta}\bar{\zeta}_\alpha\zeta_\beta\omega_1^\beta\sum_\nu\omega_\nu^\beta\bar{\omega}_\nu^\alpha$$

and by the orthogonality relations (4) we get

$$\sum_{\nu} \bar{z}_\nu z_{\nu+1} = k \cdot \sum_{\alpha} |\zeta_\alpha|^2 \omega_\alpha.$$

Taking the imaginary parts we obtain, in view of (37), the final expression

$$A = \frac{k}{2} \sum_{\alpha=0}^{k-1} |\zeta_\alpha|^2 \sin \frac{2\pi\alpha}{k}. \tag{38}$$

This remarkable formula reveals a few facts at a glance:

1. The area A is equal to the sum of the oriented areas of the $k-1$ *regular* k-gons $\zeta_1\Pi_1, \zeta_2\Pi_2, \cdots, \zeta_{k-1}\Pi_{k-1}$ into which Π is analyzed by its finite Fourier series (1). For indeed, the oriented area of the polygon $\zeta_\alpha\Pi_\alpha = (\zeta_\alpha\omega_\nu^\alpha)$ is visibly equal to

$$k \cdot \frac{1}{2} |\zeta_\alpha| \cdot |\zeta_\alpha| \sin \frac{2\pi\alpha}{k} = \frac{k}{2} |\zeta_\alpha|^2 \sin \frac{2\pi\alpha}{k},$$

which is the general term of (38). A second proof is obtained if we observe that A must reduce to the area of $\zeta_\nu\Pi_\nu$ if we assume in (38) that all $\zeta_\alpha = 0$ except ζ_ν.

2. The area A depends only on the absolute values $|\zeta_\alpha|$ of the f.F. coefficients of Π and not on their arguments.

We may finally derive an extremal property of the area. Indeed, from (38) we obtain by a now familiar argument

$$\begin{aligned} A = \frac{k}{2} \sum_{\alpha} |\zeta_\alpha|^2 \sin \frac{2\pi\alpha}{k} &\leqq \left(\max_{\alpha} \frac{1}{2} \sin \frac{2\pi\alpha}{k}\right) \cdot k \sum_{\alpha} |\zeta_\alpha|^2 \\ &= \left(\max_{\alpha} \frac{1}{2} \sin \frac{2\pi\alpha}{k}\right) \cdot \sum_{0}^{k-1} |z_\nu|^2. \end{aligned} \tag{39}$$

If we set

$$\gamma = \frac{1}{2} \max_{\alpha} \sin \frac{2\pi\alpha}{k},$$

we may conclude that

$$A \leqq \gamma \sum_{0}^{k-1} |z_\nu|^2,$$

where we have equality if and only if we have equality in (39), which is the case if $\zeta_\alpha = 0$, except for those values of α for which $\frac{1}{2}\Im\omega_\alpha = \gamma$. This proves

THEOREM 5. *Let* $\Pi = P_0P_1 \cdots P_{k-1}$ *be a closed k-gon in the oriented plane with not all vertices coalescent, O a further point in the plane. A being the oriented area of* Π, *consider the ratio*

$$R = \frac{A}{\sum (OP_\nu)^2}.$$

There are three different cases

1. *If* $k=4p$ *then*

$$R \leqq \frac{1}{2},$$

with equality if and only if our polygon is $z_\nu=\zeta_p\omega_p^\nu=\zeta_p\omega_p^\nu=\zeta_p i^\nu$, *with O at the origin, i.e.* Π *is a square of center O described p times.*

2. If $k=4p\pm1$, we have

$$R \leqq \frac{1}{2}\sin\frac{2\pi p}{k}=\frac{1}{2}\cos\frac{\pi}{2k},$$

with equality if and only if Π *is the starred k-gon* $z_\nu=\zeta_p\omega_p^\nu$, *and O is its center.*

3. *If* $k=4p+2$, *we have*

$$R \leqq \frac{1}{2}\sin\frac{2\pi p}{k}=\frac{1}{2}\cos\frac{\pi}{k},$$

with equality if and only if Π *is of the form* $z_\nu=\zeta_p\omega_p^\nu+\zeta_{p+1}\omega_\nu^{p+1}$, *with O at its center.*

In the particular cases when $k=3$ and $k=4$, we have the following statements:

Let $P_0P_1P_2$ *be a triangle, O a point in its plane, then*

$$\text{Area of } P_0P_1P_2 \leqq \frac{\sqrt{3}}{4}((OP_0)^2+(OP_1)^2+(OP_2)^2),$$

with equality if and only if the triangle is equilateral with O at its center.

Let $P_0P_1P_2P_3$ *be a quadrilateral, O a point in its plane, then*

$$\text{Area of } P_0P_1P_2P_3 \leqq \frac{1}{2}((OP_0)^2+(OP_1)^2+(OP_2)^2+(OP_3)^2),$$

with equality if and only if the quadrilateral is a square with O at its center.

9. The isoperimetric inequality for equilateral polygons. We shall now assume our polygon $\Pi=(z_0, z_1, \cdots, z_{k-1})$ to be equilateral, which means that

$$|z_{\nu+1}-z_\nu| = a, \qquad (\nu=0, \cdots, k-1).$$

If we denote by $L=ka$ its perimeter, then

$$\sum_\nu |z_{\nu+1}-z_\nu|^2 = ka^2 = L^2/k,$$

so that by (33) we obtain

$$L^2=\sum_{\nu=1}^{k-1} 4k^2\sin^2\frac{\pi\nu}{k}\cdot|\zeta_\nu|^2.$$

On the other hand, by multiplying (38) by $4k\tan(\pi/k)$ we have

$$\left(4k\tan\frac{\pi}{k}\right)A = \sum_{\nu=1}^{k-1} 4k^2 \tan\frac{\pi}{k}\sin\frac{\pi\nu}{k}\cos\frac{\pi\nu}{k}\cdot|\zeta_\nu|^2.$$

Notice that the coefficients of $|\zeta_1|^2$ in these two expansions agree; hence if we subtract these two equations, those terms cancel and we obtain

$$L^2 - \left(4k\tan\frac{\pi}{k}\right)A = \sum_{\nu=2}^{k-1} 4k^2 \sin\frac{\pi\nu}{k}\left(\sin\frac{\pi\nu}{k} - \tan\frac{\pi}{k}\cos\frac{\pi\nu}{k}\right)\cdot|\zeta_\nu|^2.$$

Since all coefficients on the right-hand side are positive, this relation proves the following last

THEOREM 6 (Due to J. Steiner). *If* $\Pi = P_0P_1\cdots P_{k-1}$ *is a plane equilateral closed k-gon of area A and perimeter L then*

$$L^2 - \left(4k\tan\frac{\pi}{k}\right)A \geqq 0, \tag{40}$$

with equality if and only if Π *is an ordinary regular k-gon.*

The inequality (40) is called isoperimetric because it implies that among all equilateral k-gons of given perimeter L (and therefore having all sides equal to L/k), the regular k-gon has the largest area. [7]

Notes and References

1. The significance of the finite Fourier series (1) for the geometry of the triangle ($k=3$) is briefly discussed in the instructive book of Ernesto Cesàro, Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung, Leipzig, 1904.

2. We denote, as usual, by $[x]$ the greatest integer not exceeding x.

3. For purposes of motivation we use here and in the next paragraph terms which are familiar from the theory of the Fourier series, a fact which of course in no way presupposes a knowledge of Fourier series theory.

4. This result may also be phrased in terms of matrices and linear transformations as follows: If we denote the linear transformations (1) and (17) by $(z)=\Omega(\zeta)$ and $(z')=A(z)$ respectively, then the matrix $\Omega^{-1}A\Omega$ assumes the diagonal form

$$\Omega^{-1}A\Omega = \begin{Vmatrix} f(\omega_0) & 0 & \cdots & 0 \\ 0 & f(\omega_1) & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & f(\omega_{k-1}) \end{Vmatrix},$$

a fact which is readily verified directly. In this form our result is substantially equivalent to a theorem of Spottiswoode on cyclic determinants; see Th. Muir, Theory of Determinants, vol. 2, p. 405. I owe this last reference to Alfred Brauer.

5. Martin Rosenman, Problem no. 3547, this MONTHLY, vol. 39, 1932, p. 239. An elegant solution was given by R. E. Huston in this MONTHLY, vol. 40, 1933, pp. 184–185.

6. Stated in terms of matrices, our question is to decide if the nth power A^n, of a given cyclic matrix A, tends to a limit as $n\to\infty$. This problem was recently solved for an arbitrary square matrix A of complex elements by R. Oldenburger, Infinite powers of matrices and characteristic roots, Duke Mathematical Journal, vol. 6, 1940, pp. 357–361, and A. Dresden, On the iteration of linear homogeneous transformations, Bulletin of the American Math. Society, vol. 48, 1942, pp. 577–579.

7. A wealth of geometric applications of the Fourier series was given by Adolf Hurwitz in his famous paper: Sur quelques applications géométriques des séries de Fourier, Mathematische Werke, Basel, vol. 1, 1932, pp. 509–554. The above given derivation of the inequality (40) by means of the finite Fourier series is due to W. Blaschke, Kreis und Kugel, Leipzig, 1916, pp. 13–20. Blaschke avoids the use of complex numbers, a fact which robs the method of much of its elegance and requires a distinction of two cases according to the parity of k.

MATHEMATICAL NOTES

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ON ALMOST PRIMES*

P. ERDÖS

1. Introduction. D. H. Lehmer [1] and others have studied odd composite numbers n which behave like primes in that they satisfy the congruence

$$2^n \equiv 2 \pmod{n}.$$

For brevity, we call such numbers almost primes. In a previous note [2] we proved that for every k there exist infinitely many square free almost primes having k distinct prime factors; this generalizes a result of Lehmer for $k \leqq 3$. In the present note we estimate from above the number of almost primes less than a given limit.

2. Theorem. Our result is the following.

THEOREM. *Let $f(x)$ denote the number of almost primes $\leqq x$. Then, for x sufficiently large, we have*

$$f(x) < x \exp\left\{-\tfrac{1}{3}(\log x)^{1/4}\right\}.$$

Remark. Since the number of primes $\leqq x$ is asymptotic to $x/\log x$, our theorem implies that the number of almost primes $\leqq x$ is very much less than the number of actual primes.

3. Proof. Let $g(n)$ be the least positive exponent e such that

$$2^e \equiv 1 \pmod{n}.$$

We separate the almost primes $n \leqq x$ into two classes C_1 and C_2. The class C_1 consists of those n's for which

$$g(n) \leqq [\exp((\log x)^{1/2})] = H,$$

while C_2 consists of all the other almost primes $\leqq x$.

The members of C_1 are divisors of

* Revised by D. H. Lehmer.

$$P = \prod_{r=1}^{H} (2^r - 1).$$

Let $q_1, q_2, \cdots, q_k$ be all the prime factors of P. Then the members of C_1 are included in the class Γ_1 of integers $\leqq x$ having prime factors taken from the set $q_1, \cdots, q_k$ only. Since 2^r-1 has less than r prime factors, we have

$$k < \sum_{r=1}^{H} r \leqq H^2. \tag{1}$$

We now separate the members of Γ_1 into two subclasses $\Gamma_{1,1}$ and $\Gamma_{1,2}$, where $\Gamma_{1,1}$ consists of those members of Γ_1 which have less than

$$W = \frac{1}{10} (\log x)^{1/2}$$

distinct prime factors. From the fact that if $m \leqq x$ and if g^α divides m, then $\alpha \leqq (\log x)/\log 2$, it follows from (1) that the number of members of $\Gamma_{1,1}$ is less than

$$(\log x/\log 2)^W \sum_{t=1}^{W} \binom{k}{t} < W k^W (\log x/\log 2)^W,$$

a quantity less than $x^{1/4}$ for all sufficiently large x.

We consider next the class $\Gamma_{1,2}$. Let $d(m)$ denote the number of divisors of m, and let $v(m)$ be the number of distinct prime factors of m. If m belongs to $\Gamma_{1,2}$, then

$$d(m) \geqq 2^{v(m)} = \exp \{v(m) \log 2\} > \exp (W/2).$$

Hence, if N is the number of members of $\Gamma_{1,2}$, we have

$$2x \log x > x \sum_{m \leqq x} m^{-1} \geqq \sum_{m \leqq x} \left[\frac{x}{m}\right] = \sum_{m \leqq x} d(m) \geqq \sum_{m \subset \Gamma_{1,2}} d(m) \geqq N \exp (W/2).$$

That is, we have

$$N \leqq 2x(\log x) \exp (-W/2).$$

Therefore, if x is sufficiently large, the total number of members of C_1 is less than

$$x^{1/4} + N \leqq x^{1/4} + 2x(\log x) \exp (-W/2) < x \exp \left\{-\frac{1}{30} (\log x)^{1/2}\right\}. \tag{2}$$

We take up now the class C_2 which we separate into two classes $C_{2,1}$ and $C_{2,2}$. The class $C_{2,1}$ consists of those members n of C_2 which have a prime factor p such that the greatest common divisor δ of $n-1$ and $p-1$ satisfies

$$\delta = (n - 1, p - 1) \geqq \exp ((\log x)^{1/4}) = T.$$

In other words, for each member n of $C_{2,1}$ there is a prime p and an integer m such that

$$n = pm, \qquad p = \delta t + 1, \qquad m = \delta u + 1, \qquad x/p \geqq m > 1.$$

The last inequality follows from the fact that n is composite. If p and δ are fixed, the number of choices for m is at most $x/(\delta p)$. Hence the number of members of $C_{2,1}$ does not exceed

$$\sum_{\delta>T} \sum_{\substack{p\leqq x \\ p\equiv\delta t+1}} x/(\delta p) < x \sum_{\delta>T} \sum_{t<x/\delta} (t\delta^2)^{-1} < x \sum_{\delta>T} 2(\log x/\delta)\delta^{-2} \tag{3}$$
$$< 2x(\log x)T^{-1} < x \exp\left\{-\tfrac{1}{2}(\log x)^{1/4}\right\},$$

for x sufficiently large.

Finally we consider the class $C_{2,2}$. This consists of almost primes

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

for which

$$\delta_i = (n-1, p_i - 1) < T, \qquad (i = 1, 2, \cdots, k).$$

It is well known that the exponent $g(n)$ divides

$$\phi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1).$$

Also $g(n)$ divides $n-1$ since n is an almost prime. Hence

$$H \leqq g(n) \leqq (n-1, \phi(n)) \leqq \prod_{i=1}^{k} (n-1, p_i - 1) \leqq T^k.$$

That is,

$$k \geqq (\log H)/\log T = \log T.$$

Thus, if M denotes the number of members of $C_{2,2}$, we have, as before,

$$2x \log x > \sum_{m\leqq x} d(m) \geqq \sum_{m\leqq x} 2^{\nu(m)} \geqq 2^k \sum_{m\subset C2,2} 1 \geqq M \cdot 2^{\log T}.$$

Hence, for x sufficiently large, we have

$$M \leqq 2x(\log x) \exp\left\{-(\log 2)\log T\right\}$$
$$\leqq x \exp\left\{-\tfrac{1}{2}(\log x)^{1/4}\right\}.$$

Combining this result with (2) and (3) we have

$$f(x) < x\left\{\exp\left(-\tfrac{1}{30}(\log x)^{1/2}\right) + 2\exp\left(-\tfrac{1}{2}(\log x)^{1/4}\right)\right\}$$
$$< x \exp\left(-\tfrac{1}{3}(\log x)^{1/4}\right),$$

for x sufficiently large. This is our theorem.

4. Discussion. By a slightly more complicated argument we could prove that, for some positive constant c,

$$f(x) < x \exp\left\{-c(\log x)^{1/2}\right\};$$

but the true order of $f(x)$ seems to be considerably smaller. As far as I know, the only estimate for $f(x)$ from below is

$$f(x) > C \log x,$$

which is due to Lehmer.

Added later. As far as I know the question of the existence of even numbers satisfying $2^n \equiv 2 \pmod{n}$ has not been considered. Except for the trivial case $n=2$, I have not succeeded in finding any such even numbers.* By the method of this paper it is easy to see that their number $\leqq x$ is certainly less than $x \exp\left\{-\frac{1}{3}(\log x)^{1/4}\right\}$.

References

1. D. H. Lehmer, On the Converse of Fermat's Theorem, I, II, this MONTHLY, vol. 43, 1936, pp. 347–354; vol. 56, 1949, pp. 300–309. These papers contain references to other work on almost primes.

2. P. Erdös, On the Converse of Fermat's Theorem, this MONTHLY, vol. 56, 1949, pp. 623–624.

ANGLES WITH RATIONAL TANGENTS**

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1. Introduction. The purpose of this note is to show that the class of angles having rational tangents, and the class of angles which are rational multiples of π, intersect only in the obvious cases.

2. Theorem. We shall establish the following result.

THEOREM. *If x is a rational multiple of π, and* tan x *is rational, then x is an integral multiple of $\pi/4$.*

3. Proof. Let $\tan x = q/p$. The theorem is trivially satisfied if $q=0$ or if $|p| = |q|$. Further, without loss of generality, x may be restricted to the first quadrant, so that p and q may be assumed positive, integral, unequal, and coprime.

If $x = m\pi/n$, then $e^{inx} = e^{-inx} = \pm 1$, or

$$(\cos x + i \sin x)^n = (\cos x - i \sin x)^n,$$

and

$$(p + iq)^n/(p^2 + q^2)^{n/2} = (p - iq)^n/(p^2 + q^2)^{n/2}.$$

* Added still later: Lehmer has just informed me that $2^{161038} \equiv 2 \bmod (161038)$, $(161038 = 2 \cdot 73 \cdot 1103)$.

** Revised by J. D. Swift.

Thus

$$(p - iq)^n = (p + iq)^n = (p - iq + 2iq)^n$$
$$= (p - iq)^n + \binom{n}{1}(p - iq)^{n-1}2iq + \cdots$$
$$+ \binom{n}{n-1}(p - iq)(2iq)^{n-1} + (2iq)^n.$$

Therefore $(p-iq)$ divides $(2qi)^n$, and p^2+q^2 divides $(2q)^{2n}$. Similarly, p^2+q^2 divides $(2p)^{2n}$ and therefore divides $(2^{2n}p^{2n}, 2^{2n}q^{2n}) = 2^{2n}$. Then $p^2+q^2=2^k$; but this is possible in positive coprime integers only when $p=q=k=1$, a contradiction.

4. Corollary. By writing tan nx as a rational function in terms of tan x, the reader may verify the following corollary.

COROLLARY: *The equations*

$$\sum_{k=0}^{a}(-1)^k\binom{n}{2k+1}x^{2k} = 0, \qquad a = \left[\frac{n-1}{2}\right], \quad n > 2,\ n \neq 4,$$

and

$$\sum_{k=0}^{b}(-1)^k\binom{n}{2k}x^{2k} = 0, \qquad b = \left[\frac{n}{2}\right], \qquad n > 2,$$

have no rational roots.

A GENERALIZATION OF GAUSS' LEMMA*

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1. Introduction. We say that a number *belongs to the first half* modulo a number n if it is congruent mod n to one of the numbers $1, 2, \cdots, [(n-1)/2]$, and that it *belongs to the second half* modulo n if it is congruent mod n to one of the numbers $[n/2]+1, [n/2]+2, \cdots, n-1$. The well-known Gauss' lemma can then be stated as follows:

A number A is a quadratic residue modulo an odd prime p if and only if an even number of the terms

$$A, 2A, 3A, \cdots, (p-1)A/2 \tag{1}$$

belongs to the second half mod p.

2. Theorem. In this note we shall prove the following result.

THEOREM: *If p is an odd prime and A is odd, then the number of terms in the sequence* (1) *which belong to the second half* mod p *is equal to the number of terms which belong to the second half* mod $2p$.

* Translated and revised by E. G. Straus.

We shall call it a *transition* whenever multiplication by A transforms a number of the sequence $1, 2, \cdots, (p-1)/2$ into a number of the second half, while we shall call it a *retention* if multiplication by A leaves such a number in the first half.

3. Proof of the theorem. We perform the following one-to-one mapping of the sequence $1, 2, \cdots (p-1)/2$ on itself:

$$\text{For} \quad 0 < x < p/4, \quad \text{we let} \quad x \to 2x.$$
$$\text{For} \quad p/4 < x < p/2, \quad \text{we let} \quad x \to p - 2x.$$

For the numbers $0<x<p/4$ we see that

$$Ax \equiv r \pmod{p}, \qquad 0 < r < p,$$

implies

$$2Ax \equiv 2r \pmod{p};$$

and hence the number of transitions mod p in the sequence $0<x<p/4$ equals the number of transitions mod $2p$ in the image of that sequence. Considering the sequence $p/4<x<p/2$, we see that

$$Ax \equiv r \pmod{p}, \qquad 0 < r < p,$$

implies

$$-2Ax \equiv 2p - 2r \pmod{2p};$$

and since $Ap \equiv p \pmod{2p}$ (it is here that the oddness of A is essential), we have

$$A(p - 2x) \equiv \begin{cases} p - 2r \pmod{2p} & \text{for} \quad 2r < p, \\ 3p - 2r \pmod{2p} & \text{for} \quad 2r > p. \end{cases}$$

Hence the number of transitions mod p in the sequence $p/4<x<p/2$ equals the number of transitions mod $2p$ in the image of that sequence. This completes the proof of our theorem.

4. The quadratic reciprocity theorem. With the help of this generalization of Gauss' lemma we can give a very simple variant to Gauss' third proof of the quadratic reciprocity theorem. We make the following preliminary remarks:

(1) Just as the evenness or oddness of the number of *transitions* determines whether or not A is a quadratic residue, so the evenness or oddness of the number of *retentions* determines whether or not $-A$ is a quadratic residue.

(2) The number $[n/k]$ is even if and only if n belongs to the first half mod $2k$.

In order to count the number of transitions and retentions mod $2p$ among the numbers $1, 2, \cdots, (p-1)/2$, we subdivide the interval $(0, p/2)$ by insert-

ing the points p/A, $2p/A$, $\cdots$, $(A-1)p/(2A)$. The numbers in the odd numbered subintervals yield retentions while those in the even numbered subintervals yield transitions. The last subinterval $((A-1)p/(2A), p/2)$ is an odd or even numbered one according as A is of the form $4n+1$ or $4n+3$. We shall do our counting so as to avoid counting this last interval. That is to say we count the transitions if A is of the form $4n+1$, and the retentions if A is of the form $4n+3$. The numbers obtained are

$$-[p/A]+[2p/A]-[3p/A]+\cdots+[(A-1)p/(2A)], \tag{2}$$

and

$$[p/A]-[2p/A]+[3p/A]-\cdots+[(A-1)p/(2A)], \tag{3}$$

respectively.

In deciding the evenness or oddness of these sums the signs are immaterial (they could all be replaced by + signs); in both cases it depends on whether an even or odd number of the terms

$$[p/A], [2p/A], \cdots, [(A-1)p/(2A)]$$

is odd. As remarked above, one of the terms $[kp/A]$ is odd if and only if kp is in the second half mod $2A$.

Hence we have the following result.

If $A=4n+1$, then A is a quadratic residue modulo the odd prime p if and only if (2) is even; that is, if and only if the number of transitions mod $2A$ in the multiplication of $1, 2, \cdots, (A-1)/2$ by p is even.

If $A=4n+3$, then $-A$ is a quadratic residue modulo the odd prime p if and only if (3) is even; that is, if and only if the number of transitions mod $2A$ in the multiplication of $1, 2, \cdots, (A-1)/2$ by p is even.

In particular, if A is an odd prime q, then the latter condition is equivalent to the statement that p is a quadratic residue mod q. Introducing the Legendre symbol (p/q), we can state this as follows:

If p, q are odd primes, then

$$\left(\frac{q}{p}\right)=\left(\frac{p}{q}\right) \quad \textit{if} \quad q=4n+1,$$

and

$$\left(\frac{-q}{p}\right)=\left(\frac{p}{q}\right) \quad \textit{if} \quad q=4n+3.$$

This latter statement is easily seen to be equivalent to the quadratic reciprocity theorem.

CLASSROOM NOTES

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A THEOREM ON THE REMAINDER OF A TAYLOR SERIES*

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Let $f(x)$ be a function for which at least the first n derivatives exist. Then,

$$(1)\quad f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + r_n$$

represents the Taylor expansion of $f(x)$ at the point $x=a$ and for an increment h. The remainder r_n may be expressed as

$$r_n = \frac{h^n}{n!}f^{(n)}(a+\alpha h) \quad \text{with} \quad 0 \leqq \alpha \leqq 1.$$

Therefore, r_n is a function of both a and h, and it may be important to know how it varies with a change of either one of these parameters. In reviewing the literature, an expression was found† for the derivative of r_n with respect to a only. However, in some recent work,‡ the sign of the derivative of r_n with respect to h was required under the condition that $f^{(n)}(x)$ always had the same sign in some interval. A theorem on this derivative was proven and is given below since there may also be other applications for it.

Since the sign of the derivative of r_n with respect to h may also depend on the sign of h, a more compact formulation of the theorem was obtained by finding the sign of the derivative of r_n with respect to h^n.

THEOREM. *If the nth derivative of a function $f(x)$ always has the same sign in some interval which includes the point $x=a$, then for any increment h within this interval, the derivative of r_n with respect to h^n is of the same sign as $f^{(n)}(x)$.*

PROOF. Equation (1) may be written as

$$(2)\quad r_n = f(a+h) - \sum_{i=0}^{n-1}\frac{h^i}{i!}f^{(i)}(a).$$

Differentiating this with respect to h

$$(3)\quad \frac{dr_n}{dh} = f'(a+h) - \sum_{i=1}^{n-1}\frac{h^{i-1}}{(i-1)!}f^{(i)}(a).$$

---

* The results presented here were obtained in connection with research sponsored by the Office of Naval Research.

† R. Courant, Differential and Integral Calculus, New Revised Edition, Vol. 1, p. 322, Nordeman Publishing Company, Inc., New York, 1942.

‡ R. Weatherston and J. V. Foa, Considerations on the selection of the mode of heat addition to a gas flowing in a pipe, Cornell Aeronautical Laboratory Report No. DB-605-A-1, May 1949.

However,

$$f'(a+h) = f'(a) + \frac{h}{1!}f''(a) + \cdots + \frac{h^{n-2}}{(n-2)!}f^{(n-1)}(a) + \frac{h^{n-1}}{(n-1)!}f^{(n)}(a+\beta h)$$

or

$$f'(a+h) = \sum_{i=1}^{n-1} \frac{h^{i-1}}{(i-1)!}f^{(i)}(a) + \frac{h^{n-1}}{(n-1)!}f^{(n)}(a+\beta h) \quad \text{with} \quad 0 \leqq \beta \leqq 1. \tag{4}$$

When equation (4) is substituted in equation (3), one obtains

$$\frac{dr_n}{dh} = \frac{h^{n-1}}{(n-1)!}f^{(n)}(a+\beta h) \tag{5}$$

and dividing equation (5) by nh^{n-1} yields

$$\frac{dr_n}{d(h^n)} = \frac{1}{n!}f^{(n)}(a+\beta h) \quad \text{with} \quad 0 \leqq \beta \leqq 1. \tag{6}$$

This proves the theorem since $f^{(n)}(x)$ was assumed to be always of the same sign in some interval.

THE PROBABILITY INTEGRAL

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The following evaluation requires only a knowledge of elementary calculus and thus with reasonable assumptions may be presented at the sophomore level.

Assuming its existence, let

$$\int_0^\infty e^{-x^2}dx = A, \quad \text{where} \quad \int_0^\infty = \lim_{h\to\infty}\int_0^h.$$

If the curve $z=e^{-y^2}$ be revolved about the Z-axis, the surface generated is $z=e^{-(x^2+y^2)}$ with volume:

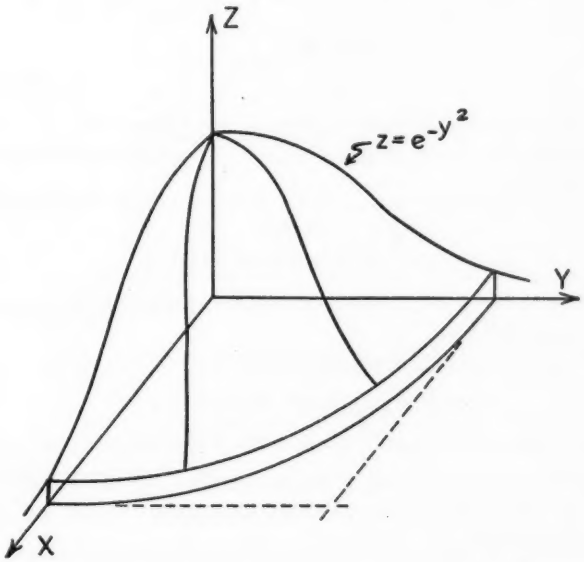
$$\begin{aligned} V &= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy \\ &= 4\int_0^\infty\left[\int_0^\infty e^{-x^2}dx\right]\cdot e^{-y^2}dy \\ &= 4\int_0^\infty A\cdot e^{-y^2}dy = 4A^2. \end{aligned}$$

But, using the method of hollow cylinders:

$$V = 2\pi \int_0^\infty ye^{-y^2}dy = \pi.$$

Thus

$$4A^2 = \pi, \qquad A = \frac{\sqrt{\pi}}{2}.$$



A DIFFERENT APPROACH TO A KNOWN INTEGRAL

H. L. Lee, University of Tennessee

The purpose of this paper is to give a new method of determining a known integral formula and to show that some integrals are special cases of this more general form. This result is obtained without the use of partial fractions or the method of completing the square as is generally the case.

Consider the integral

$$(1) \qquad \int \frac{kdu}{(a_1u + b_1)(a_2u + b_2)}$$

where $u=f(x)$ and a_1, a_2, b_1, b_2 and k are constants. It is assumed that $(a_1u+b_1)(a_2u+b_2)\neq 0$. This condition excludes the possibility that $a_i=b_i=0$ for $i=1$ or 2. There are two cases to be considered.

CASE 1. Let

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

(a) If $a_1=a_2=0$, but $b_1\neq 0$, $b_2\neq 0$, the integral (1) is of the simple form $c\int dv$.

(b) If $b_1=b_2=0$ and $a_1\neq 0$, $a_2\neq 0$, or if $a_1\neq 0$, $a_2\neq 0$, $b_1\neq 0$ and $b_2\neq 0$, (1) assumes the form $c\int v^{-2}dv$.

CASE 2. Let

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

(a) If $a_1=0$ or $a_2=0$, then (1) is of the form $c\int dv/v$.

(b) Suppose $a_1\neq 0$, $a_2\neq 0$. If $(a_1u+b_1)(a_2u+b_2)<0$, let $a_3=-a_1$ and $b_3=-b_1$, so that (1) becomes

$$-\int \frac{k\,du}{(a_3u+b_3)(a_2u+b_2)}, \tag{2}$$

where $(a_3u+b_3)(a_2u+b_2)>0$. There is therefore no loss in generality in assuming that $(a_1u+b_1)(a_2u+b_2)>0$.

With this assumption, (1) may be written

$$\frac{k}{a_1a_2}\int \frac{du}{(u+c_1)(u+c_2)}, \tag{3}$$

where $c_1=(b_1/a_1)$ and $c_2=(b_2/a_2)$. If the numerator and the denominator of (3) are divided by $(u+c_1)^2$ and the resulting integral multiplied by $-(c_2-c_1)\div -(c_2-c_1)$, (3) becomes

$$-\frac{k}{a_1a_2(c_2-c_1)}\int \frac{-\dfrac{c_2-c_1}{(u+c_1)^2}\,du}{1+\dfrac{c_2-c_1}{u+c_1}}. \tag{4}$$

When the integration of (4) is performed, the result is

$$-\frac{k}{a_1a_2(c_2-c_1)}\ln\left(1+\frac{c_2-c_1}{u+c_1}\right)+C', \tag{5}$$

which may be simplified, by replacing c_1 and c_2 by their values, into

$$\frac{k}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\ln\frac{a_1u+b_1}{a_2u+b_2}+C, \tag{6}$$

where

$$C = \frac{k}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \ln \frac{a_2}{a_1} + C'.$$

Thus the following formula is obtained

$$\int \frac{k\,du}{(a_1u + b_1)(a_2u + b_2)} = \frac{k}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \ln \frac{a_1u + b_1}{a_2u + b_2} + C. \tag{7}$$

The integral

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \frac{u - a}{u + a} + C \qquad u^2 > a^2$$

is a special case of (7) with $k=1$, $a_1=1$, $a_2=1$, $b_1=-a$ and $b_2=a$. Similarly

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \frac{a + u}{a - u} + C \qquad a^2 > u^2$$

is a special case of (7).

ON THE EQUATION OF A LINE

CAROL S. SCOTT, St. Petersburg Junior College

Given a point (x_1, y_1) and a slope m one usually writes the equation of the corresponding line in the form:

$$y - y_1 = m(x - x_1).$$

Similarly given two points (x_1, y_1) and (x_2, y_2) one finds the slope m and applies the above formula. Simplification of the resulting expression involves several algebraic steps which can be eliminated by the following method of procedure.

It is well known that for the line

$$Ax + By + C = 0$$

the slope $m=(A/-B)$. So to find the desired equation, write m in fractional form and call its numerator "A" and its denominator "$-B$." The value of C can be determined mentally from:

$$Ax_1 + By_1 + C = 0.$$

Examples: Find the equation of the line through $(4, -2)$ with slope $3/5$. Then $A=3$, $B=-5$, $C=-22$. The answer is: $3x-5y-22=0$.

Find the equation of the line through $(2, 6)$ and $(-1, 4)$. Since $m=2/3$, $A=2$ and $B=-3$. Using $(2, 6)$ we find $C=14$. As an easy check we observe that $(-1, 4)$ also satisfies the resulting equation: $2x-3y+14=0$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 921. *Proposed by R. T. Hood, Beloit College*

Devise a multiple-pan balance, with minimum number of pans, by which one can weigh any whole number of pounds using weights which are powers of n, n a given positive integer. What other amounts can be weighed with such a balance?

E 922. *Proposed by Michael Goldberg, Washington, D. C.*

Show that any given triangle can be dissected by straight cuts into four pieces which can be arranged to form two triangles similar to the given triangle.

E 923. *Proposed by R. C. Buck, University of Wisconsin*

Show that no polyhedron in three-space can have exactly seven edges, while any other integer greater than five is admissible.

E 924. *Proposed by D. J. Newman, New York University*

Find $\lim_{n\to\infty} n \sin (2\pi en!)$.

E 925. *Proposed by J. B. Kelly, University of Wisconsin*

Is it possible so to weight a pair of dice that the probability of occurrence of every sum from 2 to 12 shall be the same?

SOLUTIONS

A Cryptarithmic Greeting

E 891 [1949, 691]. *Proposed by Richard and Josephine Andree, University of Oklahoma*

Given that $E^2 = H$ and that each symbol represents a unique digit, find, and prove unique, the numerical solution of the following addition problem.

$$\begin{array}{r} D \;\&\; J \\ A\;N\;D\;R\;E\;E \\ S\;E\;N\;D \\ \hline C\;H\;E\;E\;R \end{array}$$

Solution by Esther T. Hall, College Park, Maryland. Evidently $A = 0$. Since

$E^2=H$, either $E=2$ and $H=4$, or $E=3$ and $H=9$. In the latter case $D+S+1=19$, which is impossible as D and S are different. Therefore $E=2$, $H=4$, $D+S=13$, $D+R=9$. Thus $D\neq 1, 2, 3, 4, 5, 7, 9$. From $J\neq R$ it follows that $D\neq 8$. Thus $D=6$, $R=3$, $S=7$. As $C=N+1$, $N=8$, $C=9$. The last two columns give $J=5$ and the ampersand $=1$. Consequently, the problem has at most one solution. It has at least one since the one we found checks as follows:

$$\begin{array}{cccccc} & & & 6 & 1 & 5 \\ 0 & 8 & 6 & 3 & 2 & 2 \\ & & 7 & 2 & 8 & 6 \\ \hline & 9 & 4 & 2 & 2 & 3 \end{array}$$

Also solved by Aaron Buchman, W. E. Buker, J. E. Darraugh, B. V. Dean, Monte Dernham, William Douglas, Ragnar Dybrik, J. L. Ericksen, B. K. Gold, H. E. Gould, M. S. Klamkin, F. T. Kocher, Sydney Kravitz, Elmer Latshaw, H. R. Leifer, J. J. Livers, J. B. Mertie, Jr., C. H. Murphy, J. A. Nickel, J. S. Nodvik, C. S. Ogilvy, Ingram Olkin, P. W. A. Raine, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, Nathan Schwid, Mary Seybold, Thom Sumner, C. W. Trigg, Maud Willey, and Fred Young.

In reply Dernham sent

$$MANY + HAPPY = DAYS!,$$

with $P^2=M$ and no zero, and from Trigg came

$$TRIGG + SAYS = MERCI$$

in seventeen ways, or uniquely if either $T>I>R$ or TR is an even square.

Area Property of Isogonic Configuration

E 892 [1949, 691]. *Proposed by J. P. Ballantine and G. E. Ulrich, University of Washington*

Let T be a given triangle, U the triangle whose vertices are the centroids of equilateral triangles described externally on the sides of T, and V the triangle whose vertices are the centroids of equilateral triangles described internally on the sides of T. Show that area $T=$area $U-$area V.

Solution by Roger Lessard, École Polytechnique, Montreal. It is known that triangles U and V are equilateral (*cf.*, Johnson, Mcdern Geometry, p. 221). Let A' and B' be the centroids of the equilateral triangles described externally on BC and CA, and A'' and B'' the centroids of those described internally. Then, if $BC=a$ and $CA=b$,

$$(A'B')^2 = a^2/3 + b^2/3 - (2ab/3)\cos(C + 60°),$$
$$(A''B'')^2 = a^2/3 + b^2/3 - (2ab/3)\cos(C - 60°),$$

and

$$\begin{aligned}\text{area } U - \text{area } V &= (\sqrt{3}/4)(A'B')^2 - (\sqrt{3}/4)(A''B'')^2 \\ &= (ab/2\sqrt{3})[\cos(C - 60°) - \cos(C + 60°)] \\ &= (ab/2\sqrt{3})(2 \sin C \sin 60°) \\ &= (ab \sin C)/2 = \text{area } T.\end{aligned}$$

Also solved by Joseph Carlson, F. M. Carpenter, W. B. Clarke, A. L. Epstein, D. H. Erkiletian, Jr., Albert Furman, Vern Hoggatt, L. M. Kelly, Joseph Langr, J. S. Nodvik, J. H. Simester, A. Sisk, C. W. Trigg, and the proposers.

Trigg pointed out that this problem has previously appeared as problem 358, *School Science and Mathematics*, Jan. 1914, p. 64, and problem 478, *National Mathematics Magazine*, April 1943, p. 317. For a solution by complex coordinates see problem 3367 [1930, 97].

Limit of a Series

E 893 [1949, 691]. *Proposed by N. J. Fine, University of Pennsylvania*

Find

$$\lim_{x \to 1} \sum_{n=0}^{\infty} (-1)^n \tan^{-1} x^{2n+1}.$$

Solution by W. B. Fulks, University of Minnesota. Since the series diverges for $x > 1$, the limit can exist only as a left-hand one. Then for $x < 1$ we may write

$$\begin{aligned}\sum_{n=0}^{\infty} (-1)^n \tan^{-1} x^{2n+1} &= \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{\infty} (-1)^k x^{(2n+1)(2k+1)}/(2k+1) \\ &= \sum_{k=0}^{\infty} (-1)^k/(2k+1) \sum_{n=0}^{\infty} (-1)^n x^{(2k+1)(2n+1)},\end{aligned}$$

for the double integral converges absolutely. The inner sum is easily evaluated, and our double sum becomes

$$\sum_{k=0}^{\infty} (-1)^k x^{2k+1}/(2k+1)(1 + x^{2(2k+1)}). \tag{1}$$

Designate the sum of series (1) by $S(x)$, and the kth partial sum by $S_k(x)$. Then, since the series is an alternating series with numerically decreasing terms, we have

$$|S(x) - S_k(x)| \leqq |x|^{2k+1}/(2k+1)(1 + x^{2(2k+1)}) \leqq 1/2(2k+1).$$

This establishes uniform convergence of series (1) for all real x. Hence

$$\begin{aligned}\lim_{x \to 1-0} \sum_{n=0}^{\infty} (-1)^n \tan^{-1} x^{2n+1} &= S(1) = (1/2) \sum_{k=0}^{\infty} (-1)^k/(2k+1) \\ &= (1/2) \tan^{-1} 1 = \pi/8.\end{aligned}$$

Note. For $|z|<1$, $S(z)$ represents an analytic function whose natural boundary is the unit circle. For $|z|>1$, $S(z)=S(1/z)$, and it represents another *different* analytic function whose natural boundary is also the unit circle. These are not analytic continuations of each other but they do have the same limiting values at $z=1$ and at $z=-1$. Their infinities are dense on the boundary of the unit circle.

Also solved by P. G. Kirmser, M. S. Klamkin, Roger Lessard, C. S. Ogilvy, and L. A. Ringenberg.

A Minimum Property of the Cube

E 895 [1949, 692]. *Proposed by L. Fejes Tóth, Budapest, Hungary*

Let the *incircle* of a convex polygon be defined as the largest circle whose interior may lie in the interior of the polygon. Show that the sum of the squares of the edges of a convex polyhedron is at least twelve times the square of the diameter of the least incircle of the faces.

Solution by the Proposer. Let f be the number of faces, e the number of edges, and r the radius of the least incircle of the faces of the polyhedron. Consider an arbitrary face and an arbitrary side of length s of this face, and let this side subtend an angle 2α at the center of one of the permissible positions of the incircle of the face. Then $s \geqq 2r \tan \alpha$, and the sum to be estimated is

$$S = (1/2)\sum^{2e} s^2 \geqq 2r^2 \sum^{2e} \tan^2 \alpha.$$

Applying Jensen's inequality to the function $\tan^2 \alpha$, $0 \leqq \alpha < \pi/2$, which is convex from below, we have

$$\tan^2 (\pi f/2e) \leqq \left(\sum^{2e} \tan^2 \alpha \right)/2e,$$

whence

$$S \geqq (2r)^2 e \tan^2 (\pi f/2e).$$

Since $e \leqq 3(f-2)$, this leads to

$$S \geqq (2r)^2 3(f-2) \tan^2 \{\pi f/6(f-2)\} = 3(2r)^2 T(f).$$

Numerical calculation shows that for $f=4, 5, \cdots, 13$ we have $T(f) \geqq T(6)=4$, while for $f \geqq 14$, $T(f)>12 \tan^2(\pi/6)=4$. Hence for any number of faces $S \geqq 12(2r)^2$.

Editorial Note. It is easily seen that in the relation $S \geqq 12(2r)^2$, equality holds only for the cube.

In a manner similar to the above it may be shown that if L denotes the sum of the lengths of the edges of a convex polyhedron, then $L \geqq 12\sqrt{3}r$, equality holding only for the regular tetrahedron. Fejes Tóth has established another lower bound for L. In his paper *On the total length of the edges of a polyhedron,*

Det Kongelige Norske Videnskabers Selskab Forhandlinger, Bd. XXI, Nr. 8, pp. 32–34, he has shown that $L>10D$, where D is the diameter of any sphere containing the convex polyhedron. The conjecture here is that the greatest lower bound is $12D$, the bound being attained only by the cube, but proof of this seems difficult. The conjecture has been established for convex polyhedra of equiareal faces. In the above paper Fejes Tóth has also shown that if the convex polyhedron has only triangular faces, then $L>14D$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTIONS

4370. Corrected. *Proposed by H. F. Sandham, Trinity College, Ireland*

A', B', C' are points on the opposite sides of a triangle ABC. The circles through $B'C'A$, $C'A'B$, $A'B'C$ intersect in M, the Miquel point, whose isogonal conjugate is M'. Prove that M', M are corresponding points under the direct circular transformation set up by A, A'; B, B'; C, C'.

4400. (*Originally proposed as* 3290 [1927, 491] *by J. V. Uspensky.*) *Reproposed by C. D. Olds, San Jose State College, California*

Every positive root of the equation $\tan x = x$ can be expressed as follows: $x=(p+\frac{1}{2})\pi-\theta$ where p is an arbitrary integer $\geqq 0$ and

$$\theta = C_0\,\xi + C_1\xi^3 + C_2\xi^5 + \cdots, \quad \xi = 1/(p+\tfrac{1}{2})\pi.$$

The coefficients C_0, C_1, C_2, $\cdots$, are positive rational numbers. Show that, for n very large,

$$C_n = 12^{-1/6}\Gamma(1/3)\pi^{-2/3}\cdot(\pi/2)^{2n+1}(2n+1)^{-4/3}(1+\omega_n)$$

where $\omega_n \to 0$ when $n\to\infty$, so that the above series remains convergent even for $p=0$.

4401. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In the triangle ABC for which the cotangent of the Brocard angle is equal to 2, show that the sides satisfy the relation

$$5(a^4 + b^4 + c^4) = 6(b^2c^2 + c^2a^2 + a^2b^2).$$

Show also that it is impossible for the lengths of the sides to be all integers.

4402. *Proposed by Carl Cohen, Harvard University*

Let the series of functions $\mathrm{dl}^{(n)}(x)$ be defined as

(1) $$\mathrm{dl}^{(n)}(x) = \int_0^x \mathrm{dl}^{(n-1)}(t)\,dt/t$$

(2) $$\mathrm{dl}^{(1)}(x) = -\int_0^x \log(1-t)\,dt/t \qquad \text{[Euler].}$$

Show that for $|x| < \pi/2$,

(i) $$\tan x = \sum_{n=1}^{\infty} (2/\pi)^{2n}\left[\mathrm{dl}^{(2n-1)}(1) - \mathrm{dl}^{(2n-1)}(-1)\right]x^{2n-1},$$

(ii) $$\sec x = 1 + i\sum_{n=1}^{\infty} (2/\pi)^{2n+1}\left[\mathrm{dl}^{(2n)}(-i) - \mathrm{dl}^{(2n)}(i)\right]x^{2n}.$$

4403. *Proposed by J. H. Butchart, Arizona State College*

A necessary and sufficient condition that a hyperboloid of one sheet be an altitude quadric (*i.e.* any four rulings of one regulus may be altitudes of a tetrahedron) is that any plane section perpendicular to a ruling is an equilateral hyperbola. (Compare the Proposer's note, *The altitude quadric of a tetrahedron* [1940, 383].)

4404. *Proposed by Ky Fan, University of Notre Dame*

Determine all positive solutions x of the infinite system of equations

(1) $$[[n/x]x - x] = n - 1, \qquad n = 1, 2, 3, \cdots.$$

The brackets denote, as usual, the greatest integer function.

SOLUTIONS

Infinite Series and Product

4314 [1948, 505]. *Proposed by N. J. Fine, University of Pennsylvania*

Prove the identity:

$$1 + \frac{x}{(1-x)^2} + \frac{x^2}{(1-x)^2(1-x^2)^2} + \frac{x^3}{(1-x)^2(1-x^2)^2(1-x^3)^2} + \cdots$$
$$= \frac{1 - x + x^3 - x^6 + x^{10} - \cdots}{[(1-x)(1-x^2)(1-x^3)\cdots]^2}.$$

Solution by the Proposer. Let $F(z)$ be defined by

$$(1)\quad F(z) = 1 + \frac{x}{(1-x)(1-xz)} + \frac{x^2}{(1-x)(1-x^2)(1-xz)(1-x^2z)} + \cdots,$$

where x is a fixed complex number inside the unit circle, and z is a variable $\neq x^n$, $n = 1, 2, \cdots$. Upon replacing z by zx in (1) and subtracting corresponding terms it is easy to see that

$$(2)\quad F(z) - F(zx) = \frac{zx^2}{(1-zx)(1-zx^2)} F(zx^2).$$

Now we define the function $G(z)$,

$$(3)\quad G(z) = F(z) \prod_{n \geq 1} (1 - zx^n).$$

$G(z)$ is an entire function of z, and

$$(4)\quad G(0) = F(0) = 1 + \frac{x}{1-x} + \frac{x^2}{(1-x)(1-x^2)} + \cdots = \prod_{n \geq 1} \frac{1}{1-x^n}.$$

Furthermore, from (2) and (3), we see that

$$(5)\quad G(z) - (1 - zx)G(zx) = zx^2 G(zx^2).$$

Assuming, as we may, the expansion

$$(6)\quad G(z) = a_0 + a_1 z + a_2 z^2 + \cdots,$$

where the a_n depend only on x, and equating coefficients of z^n in (5), we obtain easily

$$(7)\quad a_n = -a_{n-1}x^n \qquad n > 0,$$

the solution of which is

$$(8)\quad a_n = (-1)^n x^{(n^2+n)/2} a_0 = (-1)^n x^{(n^2+n)/2} \prod_{k \geq 1} \frac{1}{1-x^k},$$

by (4). Hence

$$(9)\quad G(z) = \prod_{k \geq 1} \frac{1}{1-x^k} \sum_{n \geq 0} (-1)^n x^{(n^2+n)/2} z^n.$$

Now, recalling (3), we have

$$(10)\quad F(z) = \prod_{k \geq 1} \frac{1}{(1-x^k)(1-x^k z)} \sum_{n \geq 0} (-1)^n x^{(n^2+n)/2} z^n.$$

Setting $z = 1$, we obtain the required result.

It should be remarked that the series in this problem completes a set of well known formulas:
Write $[0]=1$, $[n]=(1-x)(1-x^2)\cdots(1-x^n)$; then

$$(i) \qquad \sum_{n\geq 0} \frac{x^n}{[n]} = \prod_{k\geq 1} \frac{1}{1-x^k},$$

$$(ii) \qquad \sum_{n\geq 0} \frac{x^{n^2}}{[n]^2} = \prod_{k\geq 1} \frac{1}{1-x^k},$$

$$(iii) \qquad \sum_{n\geq 0} \frac{x^{n^2}}{[n]} = \prod_{k\geq 1} \frac{1}{(1-x^{5k-4})(1-x^{5k-1})},$$

$$(iv) \qquad \sum_{n\geq 0} \frac{x^n}{[n]^2} = \prod_{k\geq 1} \frac{1}{(1-x^k)^2} \sum_{m\geq 0} (-1)^m x^{(m^2+m)/2}.$$

The first two are very well known, they probably go back to Euler; *(iii)* is one of the famous Rogers-Ramanujan identities; while *(iv)* is the present problem.

In much the same way as *(iv)* we may derive

$$(v) \qquad \sum_{n\geq 0} \frac{x^{2n}}{[2n]} = \prod_{k\geq 1} \frac{1}{1-x^k} \sum_{n\geq 0} (-1)^n x^{n^2}$$

which has an interesting interpretation in terms of partition functions. If $p(m)$ is the unrestricted partition function, the right member of *(v)* can be written

$$\sum_{N\geq 0} x^N \sum_{n\geq 0} (-1)^n p(N-n^2).$$

On the other hand the left member of *(v)* is easily seen to generate the number of partitions of an integer into parts the largest of which is even, or, by reading the conjugate graph, into an even number of parts. Denoting this partition function by $p_E(N)$, we have proved

$$(vi) \qquad p_E(N) = p(N) - p(N-1) + p(N-4) - p(N-9) + \cdots.$$

It would be interesting to see if there is a combinatorial proof of *(vi)*.

Square Inscribed in Arbitrary Simple Closed Curve

4325 [1949, 39]. *Proposed by Orrin Frink, Pennsylvania State College*

Show that on every simple closed plane curve there are four points which are the vertices of a square.

Solution by C. S. Ogilvy, Trinity College, Hartford. Let m_{x_i}, or more simply, m_x, be a system of horizontal straight lines in the plane, some of which cut the curve at X_i and X_i', or more simply X and X'. Let m_{x_1} avoid the curve by passing wholly below it, and m_{x_2} avoid the curve by passing wholly above it (Jordan.) If m_x is now moved continuously from position x_1 to position x_2, it will pass across the curve, producing two continuous sequences of points of intersection

X and X'. In the event of concavities, choose those intersections which allow X and X' to move continuously along the curve. Thus there will be times when the motion of m_x and, say, X will be retrograde while accommodating the wanderings of X'; but at all times the motion of all three, and in particular the length XX', will be continuous functions.

Now let YY' be the perpendicular bisector of XX', meeting XX' in P and terminated by its intersections with the curve. There will be an initial (first contact) position of m_x where $Y'P > YP = 0$, and a final (last contact) position where $0 = Y'P < YP$. But because of the continuous behavior of XX', YY' of m_y moves continuously also, and there must be some position of YY' where $YP = Y'P$ (Bolzano). This means that there is always an inscribed rhombus whose diagonals are horizontal and vertical. It follows that there is an inscribed rhombus for any slope of the diagonals (replace the word horizontal by parallel in the first sentence).

To continue: either $XX' = YY'$, in which case the rhombus is a square and the theorem is proved; or $XX' \neq YY'$. Suppose $XX' > YY'$. Then rotate m_x continuously through 90° until it reaches the position m_y; now $XX' < YY'$. Therefore there must have been an intermediate position of the rhombus (Bolzano again) where $XX' = YY'$. This is the position of the required square.

Editorial Note. V. L. Klee refers to a paper (in Russian) by Snirelman, L. G., *On certain geometrical properties of closed curves*, Uspehi Matem. Nauk 10, 34–44 (1944). See also abstract in Math. Rev., 7, 35 (1946). The case where the curve is convex is given by Emch, *Some properties of closed convex curves in a plane*, American Journal of Mathematics, vol. 35, 1913, pp. 407–412.

Function with Natural Boundary

4329 [1949, 40]. *Proposed by A. W. Goodman, University of Kentucky*

Let θ be an irrational number, $a = e^{i\theta\pi}$. Prove that

$$f(z) = \sum_{n=0}^{\infty} a^{n^2} z^n$$

has the unit circle as a natural boundary.

Solution by R. C. Buck, Brown University. The function $f(z)$ obeys the equation

$$f(z) - 1 = azf(a^2z). \tag{1}$$

$f(z)$ has at least one singularity, z_0, on the unit circle. From (1) a^2z_0 is also a singularity; iterating this, $a^{2m}z_0$ is a singularity for $m = 0, 1, \cdots$. Since θ is irrational, these points are everywhere dense on the unit circle, which is consequently a natural boundary for $f(z)$. If θ is rational, then $f(z)$ is a rational function.

Also solved by George Piranian.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Rank Correlation Methods. By M. G. Kendall, Charles Griffin and Co. Ltd., London, 1948. 8+160 pp. 18s.

In writing this small volume the author has set a pattern that points the way towards the solution of a difficulty that is frequently encountered by persons using statistical methods in analyzing experimental data. In most publications an attempt is made to cover the whole field of statistical methods in one or two volumes. As a result many statistical methods receive such limited attention that a research worker using such a source either is unaware of the full power of the technique being considered or is not adequately informed as to the limitations and restrictions that must be considered when applying the technique. By limiting the material covered in the text to a single concept the author has been able to include most of the available theory relating to rank correlation as well as to include numerous and varied illustrative examples. Among the topics considered by the author are: the general theory of rank correlation, tied ranks, tests of significance, the problem of m rankings, partial rank correlation, and paired comparisons.

Each phase of the rank correlation problem is treated in two chapters. The first chapter is expository in nature and considers the subject from a nontechnical approach, while the second chapter contains the derivations of the formulas and the proofs of the propositions that were used in the expository chapter. Since the nontechnical chapters are entirely self-contained, the book should be of value to the research workers in applied fields especially in psychology and the social sciences. Students of statistical theory should find the book of interest since the technical chapters contain many interesting applications of the mathematical tools and devices one so often finds of use in developing the theory associated with statistical concepts.

Although the book on the whole is very well written, there are a few minor weaknesses that might be noted. The text contains an adequate number of illustrative examples; however the lack of any sets of exercises for the student affects the usefulness of the publication as a text book. One at times encounters difficulty in following the argument used in the development of the theory because of the complexity of the notation. It may be that no better notation can be developed, but one cannot help but feel that there is a lot to be gained through an improved notation. The author has confused his terminology with regard to the size of a probability value. In fact, after defining his term "small" and "very small" on page 40, he misuses the expressions in two illustrations that follow the definition.

The book is written in a style that makes it interesting reading and should find wide acceptance by workers in both the applied and theoretical field of statistics.

C. F. KOSSACK

Analytic Geometry. By A. L. Nelson, K. W. Folley, and W. M. Borgman. New York, The Ronald Press Company, 1949. 8+215 pages. $3.00.

The authors suggest that this book is to be used by undergraduates as a "preparation for the calculus rather than as a study of geometry." While one cannot separate the two concepts entirely yet it can be said that this book does satisfy the goal of the authors.

Unusual attention is given to the idea of directed line segments, which naturally make the x_2-x_1 concept more reasonable to the student. Very wise use of time and effort is devoted to the geometric meaning of the abscissa and ordinate before such topics as division of line segments, slope of a line, and the angle between two lines is discussed in the first chapter. On page 19, the figures explaining the fact that perpendicular lines have negative reciprocal slopes seem more involved than necessary even though the effort is commendable.

Throughout the book no work has been spared in making the geometric exhibits as attractive and meaningful as possible. In fact, those near the end of the book showing the ellipsoid, as well as those of the hyperboloids of both one and two sheets are excellent. Prospective calculus students will gain so much from these drawings that they should feel at home when the mechanics of the calculus is studied.

The definition of a graph (page 49) is stated with elegance. A chapter on graphs of single valued algebraic functions followed by one on single valued transcendental functions precede the study of the chapter on the circle, after which follows the conic section chapter. The reviewer is impressed with the inclusion of the topic concerning the graph of the general conic section by means of the method of the addition of two ordinates, before consideration of the transformation of coördinates is met. The latter is relegated to the last chapter just before solid analytics is reached. Since students often lose themselves in the graphing of conic sections by use of coördinate transformation methods, many instructors will welcome the arrangement of this text.

In Chapter VIII one finds appropriate emphasis given to loci and their equations. This topic is closely tied up with the usual proofs of the theorems on plane geometry by use of analytic methods.

While tables of natural and exponential functions, as well as those of natural and common logarithms appear, it would please some students and instructors to have a short résumé of algebraic and trigonometric formulae available somewhere in the text. The chapters are concise, well written, and sprinkled with stimulating graphs. Answers to odd problems are given with the problems.

The publisher seems to have done as an acceptable job as have the authors.

W. R. HUTCHERSON

Calculus. By L. L. Smail, New York, Appleton-Century-Crofts, 1949. 16+592 pages. $4.50.

It is a stated aim of the present book to pay greater attention than usual to the careful formulation of the fundamental definitions and to the meanings of the basic concepts of calculus. The author has indeed attempted to write a superior book by introducing more preciseness and rigor than is customary. To a considerable degree the author has succeeded in carrying out his laudable purpose.

One of the features of the book is the introduction of a chapter on derivatives and one on the definite integral immediately after the differentiation of algebraic functions and before the differentiation of transcendental functions. Another feature is the inclusion of carefully chosen examples illustrating concepts, definitions, and theorems as well as methods. The exercise lists, which occur at the end of almost every article, are outstanding. Answers are given in the back of the book for the odd-numbered exercises.

Professor Smail's book appears to be very teachable and should prove a welcome addition to the small class of acceptable calculus texts.

R. G. HELSEL

The Life and Works of Herbert Ellsworth Slaught. By H. J. Dark. Bureau of Publications, George Peabody College for Teachers, Nashville, Tenn. 1948. 4+152 pages, and frontispiece. $2.25.

This volume is a thoroughly documented account of the many and varied activities of Professor Slaught in educational affairs and in mathematical organizations in the United States. We find in Chapter II a brief outline of the main facts of his life. The 40 pages of Chapter III are devoted to a description of his activities and methods as a teacher, both at Peddie Institute and at the University of Chicago, including his extra-classroom activities. The mathematical texts he wrote or edited are listed in Chapter IV, together with a number of comments on their purpose and value. This chapter also includes a brief account of Slaught's work as an editor of this MONTHLY from 1907 to 1937. In Chapter V we find an extensive account of the role Slaught played in the Central Association of Science and Mathematics Teachers, the American Federation of Teachers of the Mathematical and the Natural Sciences, the American Mathematical Society, the National Committee on Mathematical Requirements, the National Council of Teachers of Mathematics, and especially in the Mathematical Association of America. This chapter is really an amazing commentary on the extraordinary vigor and vision of the man. The last two chapters are brief summaries, and there is an extensive bibliography. The many friends and admirers of Professor Slaught will find much of interest in this volume. It is written in the spirit of an enthusiastic tribute. The author naturally does not claim to have covered his subject completely, but the array of accomplishments described here is certainly inspiring, and surprising even to one who knew Professor Slaught over a number of years.

L. M. GRAVES

Freshman Mathematics. By Slobin and Wilbur. Third Edition Revised by C. V. Newsom 15+559 pages. Rinehart and Co., 1949. $5.00.

This book is a second revision of the book bearing the same title by H. L. Slobin and W. E. Wilbur. Its aim is to impart to the student "a real understanding of the fundamental principles" and appreciation of "the values of these subjects vocationally and culturally."

That this is an admirable objective is axiomatic and if a book does not achieve complete success in this respect it may still be an excellent textbook. In the humble opinion of the reviewer—humble because of his own shortcomings and doubts as a teacher—the book falls quite short of its avowed aim, as far as the first section is concerned. It is probably true that there is no unanimity among teachers as to what constitutes "fundamental concepts and principles" of freshman mathematics. Some of the most fundamental ideas are either not mentioned at all or stated very briefly or misstated. I recall hearing the late J. H. M. Wedderburn make the statement that the division transformation for polynomials is the fundamental theorem of elementary algebra; without it the work with rational algebraic expressions is hollow. It is not mentioned in the book under review. In the definition of polynomials it is stated that the exponents must be positive integers (p. 24). This is obviously a bad slip, but there is no mention of the zero polynomial and of the important fact that no degree is attached to it. It seems that the authors were trying to simplify and clarify the ideas they wanted to present and thus introduced "restricted" definitions which in some instances only confuse the issue. This is—again in the reviewer's humble opinion—regrettable particularly in case the general and correct formulation is just as simple and economical of mental effort.

The second and third sections dealing with Trigonometry and Analytic Geometry are exceptionally good. It would almost seem that they were written by a different author or authors. The material on Trigonometry is well selected and clearly presented; the solution of triangles is not over emphasized and the analytical part of the subject is given clearly and accurately so that a student can see the real value of Trigonometry.

The part dealing with Analytic Geometry is equally well done. The reviewer would like to see more time and space devoted to parametric equations, but that of course is a matter of taste. This is one of the few elementary books on the subject that discuss the problem of finding all points of intersection of two curves given in polar coördinates. The part dealing with Solid Analytic Geometry is of necessity rather brief but it is clear and very much to the point. The chapter on curve fitting is a very welcome addition to an elementary textbook and will no doubt be stimulating, both to the student and to the teacher. There is a number of tables and a bibliography that enhance the value of the book; the type is clear and I noticed only one misprint (page 71, line 8: + instead of =).

The reviewer can only hope that some day soon the first section of this book will be rewritten with the same care, good taste and discrimination that characterize the last two sections; for then it will be an outstanding book on "Freshman Mathematics."

M. S. KNEBELMAN

Klassische Funktionentheorie. Teil II. By H. Behnke and F. Sommer. Aschendorffsche Verlagsbuchhandlung, Munster Westf. 1948. 217+16 pp. DM 6.

The material of this book was prepared from lectures of H. Behnke with the editorial help of F. Sommers. The entire volume is devoted to the study of Riemann surfaces and the uniformization theory. Although the material is neither new nor complete, the presentation is in a very readable style—a style which prefers clearness to conciseness.

The book begins with a general introduction to Riemann surfaces. The postulates are made understandable by numerous examples. An analytic function having a compact Riemann surface is said to be an algebraic function. It is shown that such functions satisfy an irreducible algebraic equation and conversely. A function meromorphic on the Riemann surface belonging to an algebraic function $A(z)$ is said to belong to the function field of $A(z)$. It is shown $F(z)$ is a function of the field of $A(z)$ if and only if $F(z)$ can be expressed as a rational function of z and $A(z)$. In fact if the Riemann surface of $A(z)$ has n-sheets then the functions $1, A, \cdots, A^{n-1}$ form an algebraic basis over the field of rational functions of z for the function field of $A(z)$. The results of this theory are applied to Abelian integrals and algebraic curves.

The second half of the book is devoted to uniformization theory. A function $t=h(P)$ is called a uniformization mapping for a Riemann surface R if it is a schlicht mapping of the universal covering surface of R onto a simply connected domain D of the t-plane. The necessary machinery is first developed concerning normal families of meromorphic and schlicht functions, the Koebe distortion theorem, and universal covering surfaces. The existence of a uniformization mapping is then established and the Riemann surfaces are classified according as the normalized domain D is the unit circle, the open plane, or the closed plane. To each automorphism of the covering surface which leaves R invariant, there corresponds a bilinear transformation in the t-plane which leaves $z=h^{-1}(t)$ invariant (where z is the point of R underlying P). By definition h^{-1} is an automorphic function on D under the above group of automorphisms. A final section is devoted to automorphic functions. Here the author discusses the problem of finding a function which has a given Riemann surface for its associated Riemann surface.

R. S. PHILLIPS

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1948–1949

Mathematics Club, Adelphi College

The members of the *Mathematics Club* of Adelphi College heard three guest speakers during 1948–49. The speakers and their topics were:

Methods and models, by Col. Robert Beard of Mount Vernon, New York

Methods and models, by Lt. Col. R. C. Yates of West Point

The number e and the law of continuing growth, by Dr. Howard Fehr of Columbia University

The Hofstra College chapter of *Kappa Mu Epsilon* attended the meetings at which Colonel Beard and Dr. Fehr spoke.

Field trips were taken to the home of Rutherford Boyd at Leonia, New Jersey to see some of his artistic work and to the Brookhaven National Laboratory at Upton, New York.

The officers for 1949–50 are: President, Katherine Engler; Secretary-Treasurer, Sylvia Krasner.

The Hall Mathematics Society, Lafayette College

During the academic year 1948–49, the *Hall Mathematics Society* held six meetings. The programs for these meetings were:

The game of nim, by George Veronis

An appraisal of the life and work of Descartes, by William McLean, Professor of Mechanics

Applications of mathematics to sailing, by Karl Larsen, Professor of Physics

Electrical integrator, by Hugh Davidson, research physicist at the General Analine Corporation and founder of the idea for the integrator

Symposium on various earth maps, by George Veronis, Donald McIntyre, Robert Koch and Alexander McKittrick

The relationship of logic and mathematics, by George Clark, Professor of Philosophy.

Pi Mu Epsilon, Hunter College

The topic of the Hunter College Chapter of *Pi Mu Epsilon* for the year 1948–49 was *Highlights in the history of mathematics as reflected in some of the monumental books*. Papers presented during the first semester were:

The mathematics of ancient Babylonia, by Elaine Traub

The mathematics of ancient Egypt, by Estelle Rosen

Pythagoras and the Pythagorean School, by Helen Sarfaty

Euclid, by Sarah Kaplan

Appollonius, by Gloria Hoffman
Archimedes, by Marian Boykan
Diophantus, by Ruth Roberts
Ptolemy, by Shirley Kurman.

Papers delivered during the second semester were:

Number notations and arithmetics of ancient Babylonia and Egypt, by Arlene Withrow

Number notations and arithmetics of ancient Greece, Rome, and India, by Miriam Rosenberg

The algebra of the Hindus, by Janis Resnick
Algebra in the Arab world, by Caryl Klein
Galileo Galilei, by John Keiper
René Descartes, by Angeliki Demetriades
Gottfried Wilhelm Leibnitz, by Rita Challenger
Isaac Newton, by Josephine Guaragna.

Guest speaker at the fall initiation dinner was Prof. Carl Boyer of Brooklyn College who delivered an address entitled *The greatest analytic geometer*. At the spring initiation dinner the speaker was Prof. Richard Courant of New York University whose topic was *Existence and construction in mathematics*.

Officers for the year were: President, Josephine Guaragna; Corresponding Secretary, Estelle Rosen (1st term) and Ruth Roberts (2nd term); Recording Secretary, Rita Challenger; Treasurer, Arlene Withrow; Faculty Director, Miss Carolyn Eisele.

Mathematics Club, University of British Columbia

Papers were presented to the *Mathematics Club* of the University of British Columbia by undergraduate members at eight meetings held on the campus in the evening. The titles and authors of the papers are in the order of presentation:

Linear difference equations of the first degree, by P. C. Gilmore
Cardinal numbers, by J. L. McGregor
Theory of quaternions, by H. Harris
A proof of Minkowsky's theorem, by R. A. Macauley
Introduction to the theory of Lebesque integration, by P. Cuttle
Group theory, by G. B. Kyle
A qualitative method for some differential equations of mechanics, by P. C. Gilmore
The solution of cubic and quartic equations, by K. D. Hage.

A special speaker to the club was Professor Maslow of the department of philosophy of U.B.C. The topic of his lecture was *Mathematical logic*. Each meeting was followed by an informal discussion period while refreshments were served.

An interesting year for the club is being planned by the new president, Robin Thornton. Included in his plans are informal meetings to be held at the homes of members, adding social interest to the academic nature of the club.

Mathematics Club, Hunter College

Student papers presented at meetings of the *Mathematics Club* of Hunter College were:

Euclid, by Evelyn Margoe
Hypocycloid and epicycloid, by Patricia Cohen
Babylonian mathematics, by Elaine Traub
The birth of algebra, by Helen Smith
Center of gravity, by Marian Boykan
Stellar motions, by Flora Johns.

Faculty talks were:

Fermat's last theorem, by Prof. Marie Whelan
Mathematical induction, by Prof. Marie Whelan
The slide rule, by Prof. Hobart Bushey
The concept of curve, by Prof. Jewell H. Bushey.

The club had two supper parties during the year at the Roosevelt House, and in addition held a theatre party and a boat ride.

The officers for the year were: President, Dorothy Carli; Vice-President, Gloria Bullock; Treasurer, Helen Smith; Secretary, Doris Cohen.

Mathematics Club, United States Naval Academy

During the year 1948–49, the *Mathematics Club* of the United States Naval Academy arranged for two extra-curricular series of lectures and conferences given by members of the department of mathematics, as follows:

Modern algebra, by Prof. J. C. Abbott and Prof. T. J. Benac
Advanced calculus, by Prof. H. K. Sohl.

The club also held a series of informal evening meetings, the feature of which was a group of discussions and lectures on *The mathematical foundations of potential theory.*

For the coming year the club plans to have a larger number of guest lecturers and thus vary and supplement its program.

Pi Mu Epsilon, University of Alabama

Two university-wide competitive mathematics examinations featured the activities of the *Alabama Alpha* chapter of *Pi Mu Epsilon* this year. In the advanced section R. C. Asquith was judged the winner while W. D. Driggers was second. In the elementary division Walter Wilson won first prize and D. K. Gleason was second. In both classifications the winners received twenty dollar prizes while the runners-up won ten dollars.

Prizes were also awarded for the best student papers presented before the chapter during the year. Twenty dollars was won by Ann Lutz who spoke on *A refinement for linear interpolation* and Jack Caldwell won ten dollars for his paper entitled *The abacus*. Other papers included:

The life of Fourier, by Mrs. Charlotte Speight

Fourier Series, by Mrs. Martha Weeks
Mathematical mayhem, by Frank Butler
Plane areas by complex integration, by Prof. J. D. Mancill
Some mathematical applications of the cathode ray oscilloscope, by Robert Whitehurst.

The social activities of the chapter were high-lighted by a Christmas party and a spring picnic.

The officers for 1949–50 are: Director, Mr. Neal Rowell; Vice-Director, Frank Butler; Treasurer, Bean Green; Secretary, Dr. H. S. Thurston; Publicity Chairman, Clyde Wiley; Librarian, Dr. J. D. Mancill; Social Chairman, Miss Susie Lee Ward.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

INTERNATIONAL CONGRESS OF MATHEMATICIANS

The Association is sponsoring an exhibit of all types of mathematical models at the International Congress of Mathematicians which will be held at Harvard University on August 30–September 6, 1950. Mathematicians are encouraged to send lists of items for exhibit and requests for approximate space allotments to Lieutenant Colonel R. C. Yates, Department of Mathematics, West Point, New York.

SUMMER MEETING OF THE NATIONAL COUNCIL

The Tenth Summer Meeting of The National Council of Teachers of Mathematics will be held at The University of Wisconsin, August 21, 22, and 23. Members of the Association are welcome at all sessions, and registration fees for Association members are the same as for members of The National Council. Papers on problems of teaching mathematics in the first and second college years and on teacher education programs will be presented by H. G. Ayre, Western Illinois State College; Aaron Bakst, New York University; J. G. Bowker, Middlebury College; Henriette Brudos, North Dakota State Teachers College; C. H. Butler, Western Michigan College of Education; F. N. Fisch, Colorado College of Education; Elinor B. Flagg, Illinois State Normal University; M. L. Hartung, The University of Chicago; E. R. Heineman, Texas Technological College; J. S. McNair, Canal Zone Junior College; H. V. Price, State University

of Iowa; Vera Sanford, New York State Teachers College at Oneonta; J. H. Zant, Oklahoma A & M College.

Requests for rooms in University Residence Halls should be addressed to J. R. Mayor, North Hall, Madison 6, Wisconsin.

INSTITUTE FOR TEACHERS OF MATHEMATICS

The Institute for Teachers of Mathematics which is sponsored by the Association of Teachers of Mathematics in New England will be held at Tufts College, Medford, Massachusetts on August 22–29, 1950.

The program will include speakers on the latest developments in pure mathematics and others on applications of mathematics. There will also be discussion groups on methods of teaching and an abundance of recreation.

For more information write to Janet S. Height, Wakefield High School, Wakefield, Massachusetts or Albert Norris, Milton Academy, Milton, Massachusetts.

PERSONAL ITEMS

Dean Gillie A. Larew of Randolph-Macon Woman's College represented the Association at the inauguration of President O. W. Wake of Lynchburg College at Lynchburg, Virginia, on April 25, 1950.

Professor C. N. Moore, University of Cincinnati, served as the representative of the Association at the Convocation commemorating the Centenary of the Founding of the University of Dayton.

Professor J. R. Musselman of Western Reserve University has been appointed as the representative of the Association on the Council of the American Association for the Advancement of Science for 1950–51.

Professor H. S. M. Coxeter of the University of Toronto has been awarded the Tory Medal by the Royal Society of Canada.

Professor Henry Wallman of Massachusetts Institute of Technology who is now on leave at Chalmers Institute of Technology in Gothenburg, Sweden, has been made a foreign member of the Royal Swedish Academy.

Professor K. P. Williams of Indiana University has received a special mention award from the publishing trade for his book *Lincoln Finds a General;* also, he has been awarded a gold medal by New York University's Society of Libraries.

Dr. Iacopo Barsotti of the University of Rome and Princeton University has been appointed to an associate professorship at the University of Pittsburgh.

Professor P. F. Byrd of Fisk University has accepted a position as research scientist with the National Advisory Committee on Aeronautics.

Professor S. S. Chern of the Institute for Advanced Study has been appointed to a professorship at the University of Chicago.

Dr. Peter Chiarulli of Brown University has been appointed to an assistant professorship at Carnegie Institute of Technology.

Associate Professor Lester Dawson of Adams State College has accepted an assistant professorship at Colorado Agricultural and Mechanical College.

Dr. Werner Fenchel of the Technical University of Denmark has been appointed to a visiting professorship at the University of Southern California.

Mr. A. G. Hansen has been appointed to an instructorship at the University of Maryland.

Associate Professor R. G. Helsel who is now Acting Head of the Department of Mathematics of Ohio State University has been promoted to a professorship and named Head of the Department, effective July 1, 1950.

Dr. P. J. Kelly of the University of Southern California has received an appointment to an assistant professorship at the Santa Barbara College of the University of California.

Dr. M. S. Kramer of the University of Illinois has been appointed to an assistant professorship at the New Mexico College of Agriculture and Mechanic Arts.

Associate Professor M. Z. Krzywoblocki of the University of Illinois has been promoted to a professorship.

Dr. J. P. Nash of the Kimberly-Clark Corporation is now Research Assistant Professor at the University of Illinois.

Professor B. C. Patterson, head of the Department of Mathematics of Hamilton College, has been appointed President of Washington and Jefferson College.

Miss Ruth E. Porter of Berea College has been appointed to an assistant professorship at John Brown University, Siloam Springs, Arkansas.

Associate Professor E. L. Post, City College of the City of New York, has been promoted to a professorship.

Professor C. J. Rees who has been serving as director of graduate studies of the University of Delaware has been appointed to the first deanship of the School of Graduate Studies, effective July 1, 1950.

Assistant Professor R. W. Shephard of New York University has accepted a position with the Rand Corporation, Santa Monica, California.

Professor J. G. Van der Corput, University of Amsterdam, will spend the academic year 1950–51 at Stanford University.

Professor R. J. Walker of Cornell University has been appointed Chairman of the Department of Mathematics, effective July 1, 1950.

Assistant Professor W. R. Wasow of Swarthmore College is on leave and is serving as mathematician at the Institute for Numerical Analysis of the National Bureau of Standards.

Mr. R. T. Duffner died on March 22, 1950.

Professor Emeritus J. A. Eiesland of West Virginia University died March 11, 1950. He was a charter member of the Association.

Dr. E. D. Hellinger, visiting professor of mathematics at Illinois Institute of Technology, died on March 28, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 115 persons have been elected to membership by the Board of Governors on applications duly certified:

H. L. ALDER, Ph.D. (California) Asst. Professor, University of California, Davis, Calif.

H. W. ALEXANDER, Ph.D. (Princeton) Professor, Adrian College, Mich.

P. N. ARMSTRONG, Student, University of Nebraska, Lincoln, Nebr.

E. L. ARNOFF, M.S. (Case) Grad. Asst., California Institute of Technology, Pasadena, Calif.

LUCIEN BAGNETTO, M.S. (Louisiana) Asst. Professor, Harding College, Searcy, Ark.

JEAN M. BALDWIN, Student, Carleton College, Northfield, Minn.

JOHN BENDER, Student, Rutgers University, New Brunswick, N. J.

K. S. BERGMAN, Student, Gonzaga University, Spokane, Wash.

S. D. BERKOWITZ, Student, Columbia University, New York, N. Y.

H. F. BLASCH, Student, Hofstra College, Hempstead, N. Y.

ROBERT BREUSCH, Ph.D. (Freiburg) Asso. Professor, Amherst College, Mass.

W. E. BRIGGS, M.A. (Colorado) Part-time Instructor, University of Colordao, Boulder, Colo.

RICHARD CROWLEY CAMPBELL, A.M. (Pennsylvania) Asst. Professor, U. S. Naval Postgraduate School, Annapolis, Md.

LEONARD CARLITZ, Ph.D. (Pennsylvania) Professor, Duke University, Durham, N. C.

C. R. CARR, A.M. (Michigan) Teaching Fellow, University of Michigan, Ann Arbor, Mich.

Mona Cascio, 171 Rogers Avenue, Brooklyn, N. Y.

W. R. CASHEN, M.A. (U. of Washington) Asst. Professor, University of Alaska, College, Alaska

GEORGE CASSEB, B.S. (Chicago) Instructor, St. Mary's University, San Antonio, Texas.

T. E. CHEATHAM, JR., Student, Purdue University, Lafayette, Ind.

S. S. CHERN, D.Sc. (Hamburg) Professor, University of Chicago, Ill.

A. J. COLEMAN, Ph.D. (Toronto) Lecturer, University of Toronto, Ont.

ESTHER COMEGYS, Ph.D. (Radcliffe) Asst. Professor, University of Maine, Orono, Maine.

JOANN M. CUMMING, B.S. (Mt. St. Vincent) Grad. Asst., Marquette University, Milwaukee, Wis.

K. B. CUMMINS, M.A. (Bowling Green State) Teacher, New Washington High School, Ohio.

J. F. DARLING, A.B. (Cornell) 11 W. Grant St., Woodstown, N. J.

L. C. DAWSON, M.A. (Wichita) Asst. Professor, Colorado A. and M. College, Fort Collins, Colo.

A. S. DAY, Ph.D. (Yale) Asst. Professor, Wesleyan University, Middletown, Conn.

LUIS DE GREIFF B., Civ.Eng. (Universidad Nacional de Colombia) Professor, Facultad de Minas, Medellin, Colombia.

ANNIE L. DICKS, A.M. (South Carolina) Professor, Columbia College, S. C.

R. J. DICKSON, JR., B.S. (Stanford) Grad. Assistant, California Institute of Technology, Pasadena, Calif.

A. R. DIDONATO, Student, Duquesne University, Pittsburgh, Pa.

RAGNAR DYBVIK, Cand.Real (Oslo) Lektor, Levanger kommunale hogre almenskole, Norway.

R. D. EDWARDS, M.A. (Pittsburgh) Instructor, University of Pittsburgh, Pa.

P. G. ENGSTROM, Student, Bethany College, Lindsborg, Kansas.

H. D. FELDMAN, Student, Rutgers University, New Brunswick, N. J.

E. R. FINCK, JR., Student, St. Mary's University, San Antonio, Texas.

R. M. FREEBORN, A.B. (Whittier Coll.) Instructor, Mark Keppel High School, Alhambra, Calif.

Leona Freeman, B.A. (Hunter Coll.) Teacher, Stitt Jr. High School, New York, N. Y.

C. B. GERMAIN, Student, College of St. Thomas St. Paul, Minn.

LEON GOLDSTEIN, Staff Member, Division of Industrial Cooperation, Massachusetts Institute of Technology, Cambridge, Mass.

W. A. GOLOMSKI, M.S. (Marquette) Grad. Asst., Marquette University, Milwaukee, Wis.

CLEM GRABNER, JR., M.A. (Indiana) Asst. Professor, Madison College, Harrisonburg, Va.

EMIL GROSSWALD, M.S. (Bucharest) Asst. Instructor, University of Pennsylvania, Philadelphia, Pa.

L. C. HILL, B.S. in Ed. (Miami U.) Grad. Student, Miami University, Oxford, Ohio.

HILDEGARDE HORENI, M.A. (Minnesota) Instructor, State Teachers College, Mankato, Minn.

R. J. JAEGER, JR., Student, Hofstra College, Hempstead, N. Y.

P. B. JOHNSON, Ph.D. (C.I.T.) Asst. Professor, Occidental College, Los Angeles, Calif.

L. G. JONES, M.A. (Oregon) Grad. Asst., University of Oregon, Eugene, Ore.

JOHN KAISER, M.A. (Ohio U.) Asso. Professor, Kent State University, Ohio.

RAYMOND KASSLER, Student, Brooklyn College, N. Y.

J. B. KELLY, Ph.D. (M.I.T.) Instructor, University of Wisconsin, Madison, Wis.

T. W. KELLY, B.S. (C.I.T.) Project Engineer, Holloman Air Force Base, Alamogordo, N. M.

MRS. EVELYN K. KINNEY, M.A. (Illinois) Head of Math. Dept., Ferrum Junior College, Va.

J. N. P. LAWRENCE, Student, Johns Hopkins University, Baltimore, Md.

L. H. N. LEE, M.S. (Minnesota) Instructor, University of Minnesota, Minneapolis Minn.

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DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The regular fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at American University, Washington, D. C., on Saturday, December 10, 1949. Professor L. M. Kells, the Chairman of the Section, presided at both the morning and afternoon sessions.

Sixty-five persons registered their attendance at the meeting, including the following forty-six members of the association: R. P. Bailey, R. H. Bing, W. E. Bleick, R. S. Burington, H. H. Campaigne, Randolph Church, G. F. Cramer, C. H. Frick, B. C. Getchell, Michael Goldberg, R. A. Good, E. C. Gras, D. W.

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The spring meeting of the section will be held during May, 1950. The place and exact date will be announced at a later time.

The following papers were presented:

1. *A Sturmian separation theorem*, by Professor C. Gras, United States Naval Academy.

Consider $\Phi=\phi_1 u+\phi_2 Ku'$ and $\Psi=\psi_1 u+\psi_2 Ku'$, where u is a solution of the differential equation $d(Ku')/dx=Gu$. The properties of the zeros of Φ and Ψ are readily obtained by use of vector algebra, rather than resorting to the Ricatti differential equation.

2. *Nonlinear transformations of divergent and slowly convergent sequences*, by Mr. Daniel Shanks, Naval Ordnance Laboratory, introduced by the Secretary.

An analogy between mathematical sequences and the transients of linear systems is developed. Through each $2K+1$ consecutive values of the sequence A_n one passes a continuous function of the form $B+\sum_{i=1}^{k} a_i e^{\alpha_i n}$. The series of exponentials either converges to, diverges from, or is asymptotic to, the constant B. An explicit formula for B in terms of the A_n is given, and this forms the basis of several nonlinear sequence-to-sequence transforms $A_n \to B_n$. The first order transform ($K=1$) is

$$B_n = \frac{A_n^2 - A_{n+1}A_{n-1}}{2A_n - (A_{n+1} - A_{n-1})}.$$

If iterated, this transform is very useful for obtaining numerical results. The transforms are applied to a variety of convergent and divergent sequences. Some theorems concerning these transforms, and some relations to the Padé Table, Thiele's Reciprocal Differences, and Gauss' Numerical Integration, are indicated.

3. *Mathematical requirements for the personnel of a computing laboratory*, by Dr. R. F. Clippinger, Aberdeen Proving Ground, introduced by the Secretary.

In order to operate the modern computing machine for maximum output, a staff of perhaps twenty mathematicians of varying degrees of training is required. There is currently such a shortage of persons trained for this work, that machines are not working full time. With the number of machines building or projected it is probable that within ten years, two thousand persons will be required in this work. Since this is a substantial fraction of all professional mathematicians in the country, Dr. Clippinger urged increased attention to the phases of mathematics which are essential for such work. This would include, among other topics, emphasis on differential equations, particularly general existence theorems, numerical analysis, and such algebraic topics as matrices and the inversion of matrices of high order. It was also urged that work in coding, *i.e.* specific discussion of techniques of preparation of problems for machines, be introduced. A course of this nature is currently being conducted at the Aberdeen Proving Ground.

4. *On the minimum in a sequence of random variables*, by Dr. M. L. Juncosa, The Johns Hopkins University, introduced by the Secretary.

Let $\xi_n=\min_{k\leq n} X_k$, $n=1, 2, \cdots$, where $\{X_k\}$ are mutually independent random variables

whose respective distribution functions are $\{F_k(x)\}$. Previous work of Gnedenko (*Annals of Math.*, 1943) on identically distributed X_k's is extended to X_k's with different distributions. With some natural restrictions on $\{F_k(x)\}$ some necessary and sufficient conditions for the stability and also relative stability of $\{\xi_n\}$ are given. The limiting distribution of $(\xi_n-a_n)/b_n$ where $\{a_n>0\}$ and $\{b_n\}$ are suitable sequences of constants, is investigated. The chief result is that limiting distributions much more general than Gnedenko's three classes now appear. The limiting distribution in case $F_k(x)=F(\alpha_k x+\beta_k)$ is also examined. Finally, in the case of positive random variables X_k, conditions on $\{F_k(x)\}$ are given in order for the limiting distributions of ξ/a_n to be of the form $1-e^{-x}$. (This work appears in *The Duke Mathematical Journal*, Dec. 1949.)

5. *Roulettes in gas dynamics*, by Prof. M. H. Martin, University of Maryland. This was an invited address.

The classical construction due to Prandtl and Meyer for supersonic flow of a polytropic gas around a corner is extended from isentropic flows to anisentropic flows. For isentropic corner flow, it is well known that the hodograph of the flow is an epicycloid. This epicycloid is generated by rolling a circle of radius s externally on the circle $q=r$ where $r=c$, $s=(1-c)/2$, $c=\sqrt{\gamma-1}/\sqrt{\gamma+1}$ = critical speed.

For anisentropic flow the hodographs are generated by rolling a circle of radius $|s|$ on a circle of radius r, and are described by a point fixed on a radius of the rolling circle at a distance t from its center. Here

$$r=\frac{1+q_0}{q_0+c^2}c^2, \qquad s=\frac{(1+q_0)(q_0-c^2)}{2(q_0+c^2)}, \qquad t=\frac{1-q_0}{2},$$

where q_0 is a parameter restricted to the unit interval $0<q_0<1$. Depending on the value selected for q_0, the hodograph will be a hypotrochod, circle, epitrochoid, epicycloid, or limaçon. In particular, for $0<q_0<c$, the hodograph curve crosses the sonic circle $q=c$, and the corner flow in the physical plane affords an example of transonic flow.

S. B. JACKSON, *Secretary*

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirtieth regular meeting of the Southern California Section of the Mathematical Association of America was held at Immaculate Heart College, Hollywood, California on Saturday, March 11, 1950. Professor H. R. Pyle, Chairman of the Section, presided.

The attendance was one hundred and six, including the following seventy-one members of the Association: L. J. Adams, O. W. Albert, E. F. Beckenbach, Lulu Bechtolsheim, May M. Beenken, Clifford Bell, C. M. Bell, Jr., H. F. Bohnenblust, Herbert Busemann, W. D. Cairns, L. M. Coffin, Myrtie Collier, F. E. Cothran, P. H. Daus, R. P. Dilworth, D. C. Duncan, Arthur Erdélyi, Ruth A. Fish, W. B. Fulks, S. I. Gass, B. K. Gold, Jr., J. W. Green, C. J. A. Halberg, Jr., A. F. Herbst, R. B. Herrera, M. R. Hestenes, R. E. Horton, J. M. Howell, D. H. Hyers, C. G. Jaeger, P. B. Johnson, G. R. Kaelin, Samuel Karlin, P. J. Kelly, A. J. Kempner, Cornelius Lanczos, Ella E. Lausman, L. C. Lay, Margaret B. Lehman, Fred Marer, D. W. Matlack, G. F. McEwen, F. R. Morris, W. H. Myers, J. A. S. Neilson, J. B. Nelson, D. J. Peterson, W. T. Puckett, H. R. Pyle, R. L. Rauch, E. C. Rex, J. M. Robb, R. M. Robinson, R. W. Shephard, Sister Rose Gertrude, Samuel Skolnik, I. S. Sokolnikoff, D. V. Steed, Robert Steinberg, J. D. Swift, T. E. Sydnor, W. I. Thompson, V. C. Throck-

morton, C. W. Trigg, D. D. Wall, Jean B. Walton, R. L. White, A. L. Whiteman, B. R. Wicker, R. L. Wilder, G. M. Wing.

At the business meeting the following officers were elected for the next academic year: Chairman, Herbert Busemann, University of Southern California; Vice Chairman, W. H. Glenn, Jr., John Muir College; Program Committee, R. E. Horton (Chairman), P. J. Kelly, P. A. White, and the Acting-Secretary W. T. Puckett, ex-officio; Acting-Secretary, W. T. Puckett, University of California at Los Angeles. The next meeting was scheduled for March 10, 1951 at Whittier College, Whittier, California.

The following program was presented:

1. *Models useful for mathematical instruction*, by Mrs. Margaret B. Lehman, University of California at Los Angeles, Mr. D. W. Matlack, North American Aviation, Inc., Mr. C. J. A. Halberg, Pomona College, and Mr. L. C. Lay, John Muir College.

The models exhibited were left on display during the social hour which preceded the program, and during the noon intermission. Mrs. Lehman stressed the desirability of making models with visibility in mind. Mr. Matlack exhibited twenty models representing various stellated forms of the rhombic triacontahedron, and thread models of the tangent and binormal surfaces of the "baseball curve" lying on a sphere. Mr. Halberg stressed certain aspects of the construction of models using the newer plastics. Mr. Lay exhibited a number of slides constructed by students in classes in analytic geometry.

2. *Problems sections in American mathematical magazines*, by Mr. C. W. Trigg, Los Angeles City College.

The speaker discussed the problem departments of this MONTHLY, *School Science and Mathematics*, and *Mathematics Magazine*, with particular reference to their composition, contributors, recurrent problems, and unsolved problems. He distributed mimeographed copies of unsolved problems from this MONTHLY and *Mathematics Magazine*, and urged the listeners to join the ranks of "problem solvers."

3. *Convexity and related notions*, by Professor J. W. Green, University of California at Los Angeles.

In this invited hour address, the definitions and some fundamental properties of convex sets and convex functions were outlined, and a number of generalizations of these notions were discussed, together with some of the problems arising under these definitions. Considerable interest was shown in the subject, and the discussion lasted for some time after the talk was given.

4. *Fixed points*, by Professor R. L. Wilder, University of Michigan.

Professor Wilder gave an exposition of theorems and problems concerning fixed points, and their applications, which presupposes no knowledge of topology.

5. *Quadratic congruences and quadratic reciprocity*, by Professor A. L. Whiteman, University of Southern California.

Making use of Gauss sums, the speaker presented a simple derivation of the classical formula for the number of solutions of the congruence

$$x_1^2 + x_2^2 + \cdots + x_n^2 \equiv a \pmod{p}.$$

Lebesgue's proof of the law of quadratic reciprocity was given as an application.

6. *Asymptotic series*, by Professor Arthur Erdélyi, California Institute of Technology.

In this expository paper Euler's series $1-1!x+2!x^2-3!x^3+\ldots$ was taken as a point of departure for a brief introduction to semiconvergent and asymptotic series. The speaker contrasted the operational and arithmetic points of view, and showed how such series can be made useful.

7. *The M.A. degree in mathematics as offered at San Jose State College*, by Dr. W. H. Myers, San Jose State College.

The speaker announced an extension of the graduate program at San Jose State College to include the granting of the M.A. degree in mathematics. This accredited college, whose enrollment exceeds 8000, has been on the approved list of the Association of American Universities since before the war. The fifth year of work leading to the California General Secondary Teaching Credential is already available here.

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P. H. DAUS, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
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IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Duluth Branch of University of Minnesota, October 7, 1950.
MISSOURI, Central College, Fayette, Spring, 1951.
NEBRASKA
NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, University of Washington, Seattle, June 16, 1950.
PHILADELPHIA, Lehigh University, Bethlehem, Pennsylvania, November 25, 1950.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.
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